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# Beyond perturbation 1: De Rham spaces

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ABSTRACT

It is shown that if one uses the notion of $\infty$-nilpotent elements due to Moerdijk and Reyes, instead of the usual definition of nilpotents to define reduced $C^\infty$-schemes, the resulting de Rham spaces are given as quotients by actions of germs of diagonals, instead of the formal neighbourhoods of the diagonals.



## 1. Introduction

As it is known [1], in algebraic geometry one can define connections, differential operators etc. without ever mentioning derivatives or any kind of limiting procedure. One only needs to have the notion of infinitesimal neighbourhoods given by nilpotent elements.

Looking at what happens when one contracts such neighbourhoods, one arrives at objects that have some prescribed behaviour along these contractions. The results of contractions are usually called de Rham spaces, and the objects that live on them are called crystals. Choosing to work with linear objects (i.e. sheaves of modules) one arrives at $D$-modules and linear differential operators (see e.g. [2]).

This technique can be applied also to differential geometry, once we use the theory of $C^\infty$-rings to take an algebraic–geometric approach (e.g. [3]). However, $C^\infty$-rings are much more than just commutative $\mathbb{R}$-algebras, and there is more than one way to define crystals in differential geometry because there is more than one notion of nilpotence.

It was observed in [4] that apart from the usual nilpotent elements, $C^\infty$-rings can have $\infty$-nilpotent ones, which are defined as follows: $a \in A$ is $\infty$-nilpotent, if the $C^\infty$-ring $A\{a^{-1}\}$ obtained by inverting $a$ is 0. Here it is important that this inverting happens *in the category of* $C^\infty$*-rings*. For example $x \in C^\infty(\mathbb{R})/(e^{-\frac{1}{x^2}})$ is $\infty$-nilpotent, but *not* nilpotent. Clearly every nilpotent element is also $\infty$-nilpotent.

Using the usual notion of nilpotence one gets de Rham spaces that can be described as quotients by actions of formal neighbourhoods of the diagonals. This is true in both algebraic and differential geometry. In the differential-geometric literature, however, one usually talks about jet bundles, instead of the formal neighbourhood of the diagonal and sheaves of modules over such neighbourhoods. The difference is in name only.

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Something completely different happens when we apply de Rham space formalism to contracting $\infty$-nilpotent neighbourhoods. Instead of quotients by actions of formal neighbourhoods of diagonals we get quotients by actions of *germs* of diagonals. As in the formal case, these $C^\infty$-rings of germs carry a linear topology given by order of vanishing at the diagonal. Since in differential geometry there are many more orders of vanishing than just the finite ones, the infinitesimal theory we get here is much richer.

An immediate benefit of this richer theory is having many more differential operators, than just the polynomial ones. Another consequence becomes apparent when we consider the opposite procedure—summation. Instead of the usual Taylor series, that characterize behaviour of functions in relation to algebraic monomials, we need to work with functions having arbitrary vanishing properties, i.e. we need to go to trans-series (e.g. [5]) and beyond.

Consider the deformation theory one gets from this: in addition to specifying behaviour of functors with respect to nilpotent extensions, we should consider also $\infty$-nilpotent extensions. Since all $\infty$-nilpotent extensions add up to germs, the deformation theory necessarily goes beyond perturbations. In turn this leads to a completely new definition of derived geometry. Differential graded manifolds or simplicial $C^\infty$-rings will not suffice anymore, since decomposition according to degree/simplicial dimension reflects decomposition according to finite orders of vanishing.

The present paper is the first in a series examining this rich infinitesimal theory and possibly some of its applications.

Here are the contents of the paper:

In Section 2.1 we consider 6 different radicals of ideals in $C^\infty$-rings. Three of them are well known in commutative algebra (nilradical, Jacobson radical, and intersection of all maximal ideals having $\mathbb{R}$ as the residue field), while another two come from considering different Grothendieck topologies on the category of $C^\infty$-spaces. The main actor of this paper – the $\infty$-radical – is specific to the $C^\infty$-algebra.

In Section 2.2 we prove that just as nilradical satisfies the strong functoriality property, so does the $\infty$-radical. We give proofs for some useful facts relating to algebra of these radicals, preparing the ground for de Rham spaces.

Before we can define de Rham spaces for both nilpotent and $\infty$-reductions we introduce regularity conditions in terms of injectivity with respect to the reduction functors. This is done in Section 3.1.

Then in Section 3.2 we define the two kinds of de Rham spaces. Here we also prove the central result that these de Rham spaces can be built using certain neighbourhoods of the diagonals. The main ingredient in the proof is the strong functoriality from Section 2.2.

Section 3.3 is dedicated to presenting these neighbourhoods of the diagonals as spectra of $C^\infty$-rings equipped with linear topologies. In the nilpotent case these spectra are just the usual formal neighbourhoods, while in the $\infty$-case they are the germs of diagonals.

De Rham groupoids appear in Section 3.4. We show in particular that both in the nilpotent and the $\infty$-cases the de Rham spaces we construct are weakly equivalent (as simplicial sheaves) to the nerves of the corresponding de Rham groupoids. We show that often (e.g. for all manifolds) these nerves consist of the formal neighbourhoods and respectively of the germs of diagonals in all Cartesian powers.

Finally in Section 4 we look at the differential operators one obtains from $\infty$- de Rham groupoids. We postpone a detailed analysis to another paper, but we do show that these operators go beyond perturbation, i.e. they provide infinitesimal description of deformations of the identity morphism on a manifold whose infinite jets at the identity vanish.

## 2. Radicals and reductions in $C^\infty$-algebra

Let $\mathrm{C^\infty R}$ be the category of $C^\infty$-rings, we denote by $\mathrm{C^\infty R_{fg}} \subset \mathrm{C^\infty R}$ the full subcategory of finitely generated $C^\infty$-rings. By definition (e.g. [3] §I.1) $\mathrm{C^\infty R}$ is the category of product preserving functors $\mathfrak{R}_\infty \to \mathrm{Set}$, where $\mathfrak{R}_\infty$ is the algebraic theory of $C^\infty$-rings, i.e. it is the category having $\{\mathbb{R}^n\}_{n\in\mathbb{Z}_{\geq 0}}$ as objects and $C^\infty$-maps as morphisms. Objects of $\mathrm{C^\infty R_{fg}}$ are quotients of $\{C^\infty(\mathbb{R}^n)\}_{n\geq 0}$ by ideals ([3], §I.5).

Geometric constructions are usually performed in $\mathrm{C^\infty R_{fg}}$ (e.g. [3]), with infinitely generated $C^\infty$-rings recovered as Ind-objects in $\mathrm{C^\infty R_{fg}}$. However, sometimes it is necessary to consider infinitely generated $C^\infty$-rings directly. For example, there are many ideals in $C^\infty(\mathbb{R}^n)$, that are not finitely generated, consequently simplicial resolutions of $C^\infty$-rings often have components that are not finitely generated. This becomes important, for example, in [6]. In this paper we try to work in all of $\mathrm{C^\infty R}$, and switch to $\mathrm{C^\infty R_{fg}}$ only when necessary.

The category $\mathrm{C^\infty R}$ is both complete and co-complete (e.g. [7] Cor. 1.22, Thm. 4.5). We denote the coproduct in this category by $\overline{\otimes}$. For a $C^\infty$-ring $A$ we denote by $\mathrm{Spec}(A)$ the corresponding object in the opposite category $\mathrm{C^\infty R^{op}}$. And given $\mathcal{X} \in \mathrm{C^\infty R^{op}}$ we write $C^\infty(\mathcal{X})$ for the corresponding object of $\mathrm{C^\infty R}$. For an arbitrary $\mathcal{X} \in \mathrm{C^\infty R^{op}}$ we denote by $\underline{\mathcal{X}}$ the representable pre-sheaf $\mathrm{hom}_{\mathrm{C^\infty R^{op}}}(-, \mathcal{X}) : \mathrm{C^\infty R^{op}} \to \mathrm{Set}$.

### 2.1. Six radicals

Let $A \in \mathrm{C^\infty R}$ be a $C^\infty$-ring, as $\mathfrak{R}_\infty$ contains the theory $\mathfrak{R}_{alg}$ of commutative, associative, unital $\mathbb{R}$-algebras, every $\mathfrak{R}_\infty$-congruence on $A$ is given by an ideal of the underlying commutative $\mathbb{R}$-algebra. The converse is also true (e.g. [3], prop. I.1.2)