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Clifford coherent state transforms on spheres

Pei Dang [a], José Mourão [b], João P. Nunes [b,*], Tao Qian [c]

[a] *Faculty of Information Technology, Macau University of Science and Technology, Macao*
[b] *Department of Mathematics and Center for Mathematical Analysis, Geometry and Dynamical Systems, Instituto Superior Técnico, University of Lisbon, Portugal*
[c] *Department of Mathematics, Faculty of Science and Technology, University of Macau, Macao*

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ABSTRACT

We introduce a one-parameter family of transforms, $U^t_{(m)}$, $t > 0$, from the Hilbert space of Clifford algebra valued square integrable functions on the m–dimensional sphere, $L^2(\mathbb{S}^m, d\sigma_m)\otimes\mathbb{C}_{m+1}$, to the Hilbert spaces, $\mathcal{M}L^2(\mathbb{R}^{m+1}\setminus\{0\}, d\mu_t)$, of solutions of the Euclidean Dirac equation on $\mathbb{R}^{m+1}\setminus\{0\}$ which are square integrable with respect to appropriate measures, $d\mu_t$. We prove that these transforms are unitary isomorphisms of the Hilbert spaces and are extensions of the Segal–Bargman coherent state transform, $U_{(1)} : L^2(\mathbb{S}^1, d\sigma_1) \longrightarrow \mathcal{H}L^2(\mathbb{C}\setminus\{0\}, d\mu)$, to higher dimensional spheres in the context of Clifford analysis. In Clifford analysis it is natural to replace the analytic continuation from $\mathbb{S}^m$ to $\mathbb{S}^m_{\mathbb{C}}$ as in (Hall, 1994; Stenzel, 1999; Hall and Mitchell, 2002) by the Cauchy–Kowalewski extension from $\mathbb{S}^m$ to $\mathbb{R}^{m+1}\setminus\{0\}$. One then obtains a unitary isomorphism from an L^2–Hilbert space to a Hilbert space of solutions of the Dirac equation, that is to a Hilbert space of monogenic functions.

1. Introduction

In this work, we continue to explore the extensions of coherent state transforms to the context of Clifford analysis started in [1–4]. In [3], an extension of the coherent state transform (CST) to unitary maps from the spaces of L^2 functions on $M = \mathbb{R}^m$ and on the m–dimensional torus, $M = \mathbb{T}^m$, to the spaces of square integrable monogenic functions on $\mathbb{R} \times M$ was studied.

We consider the cases when M is an m–dimensional sphere, $M = S^m$, equipped with the $SO(m+1,\mathbb{R})$–invariant metric of unit radius. These cases are a priori more complicated than those studied before as the transform uses (for $m > 1$) the Laplacian and the Dirac operators for the non–flat metrics on the spheres. We show that there is a unique $SO(m+1,\mathbb{R})$ invariant measure on $\mathbb{R} \times \mathbb{S}^m \cong \mathbb{R}^{m+1} \setminus \{0\}$ such that the natural Clifford CST (CCST) is unitary. This transform is factorized into a contraction operator given by heat operator evolution at time $t = 1$ followed by Cauchy–Kowalewski (CK) extension, which exactly compensates the contraction for our choice of measure on $\mathbb{R}^{m+1} \setminus \{0\}$. In the usual coherent state Segal–Bargmann transforms [5–10], instead of the CK extension to a manifold with one more real dimension, one considers the analytic continuation to a complexification of the initial manifold (playing the role of phase space of the system). The CCST is of interest in Quantum Field Theory as it establishes natural unitary isomorphisms between Hilbert spaces of solutions of the Dirac equation and one-particle Hilbert spaces in the Schrödinger representation. The standard CST, on the other hand, studies the unitary equivalence of the Schrödinger representation with special Kähler representations with the wave functions defined on the phase space.

In the Section 3.2 we consider a one-parameter family of CCST, using heat operator evolution at time $t > 0$ followed by CK extension, and we show that, by changing the measure on $\mathbb{R}^{m+1} \setminus \{0\}$ to a new Gaussian (in the coordinate $\log(|\underline{x}|)$)

* Corresponding author.
E-mail address: jpnunes@math.tecnico.ulisboa.pt (J.P. Nunes).

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measure $d\mu_t$, these transforms are unitary. As t approaches 0 (so that the first factor in the transform is contracting less than for higher values of t) the measures $d\mu_t$ become more concentrated around the radius $|\underline{x}| = 1$ sphere and as $t \to 0$, the measure $d\mu_t$ converges to the measure

$$\delta(y)\, dy\, d\sigma_m\,,$$

where $y = \log(|\underline{x}|)$, supported on $\mathbb{S}^m$.

2. Clifford analysis

Let us briefly recall from [11–18], some definitions and results from Clifford analysis. Let $\mathbb{R}_{m+1}$ denote the real Clifford algebra with $(m+1)$ generators, $e_j, j = 1, \ldots, m+1$, identified with the canonical basis of $\mathbb{R}^{m+1} \subset \mathbb{R}_{m+1}$ and satisfying the relations $e_i e_j + e_j e_i = -2\delta_{ij}$. Let $\mathbb{C}_{m+1} = \mathbb{R}_{m+1} \otimes \mathbb{C}$. We have that $\mathbb{R}_{m+1} = \oplus_{k=1}^{m+1} \mathbb{R}^k_{m+1}$, where $\mathbb{R}^k_{m+1}$ denotes the space of k-vectors, defined by $\mathbb{R}^0_{m+1} = \mathbb{R}$ and $\mathbb{R}^k_{m+1} = \text{span}_{\mathbb{R}}\{e_A \,:\, A \subset \{1, \ldots, m+1\}, |A| = k\}$, where $e_{i_1 \ldots i_k} = e_{i_1} \ldots e_{i_k}$.

Notice also that $\mathbb{R}_1 \cong \mathbb{C}$ and $\mathbb{R}_2 \cong \mathbb{H}$. The inner product in $\mathbb{R}_{m+1}$ is defined by

$$\langle u, v \rangle = \left(\sum_A u_A e_A, \sum_B v_B e_B \right) = \sum_A u_A v_A.$$

The Dirac operator is defined as

$$\underline{D} = \sum_{j=1}^{m+1} e_j\, \partial_{x_j}.$$

We have that $\underline{D}^2 = -\Delta_{m+1}$.

Consider the subspace of $\mathbb{R}_{m+1}$ of 1-vectors

$$\{\underline{x} = \sum_{j=1}^{m+1} x_j e_j :\; x = (x_1, \ldots, x_m) \in \mathbb{R}^{m+1}\} \cong \mathbb{R}^{m+1},$$

which we identify with $\mathbb{R}^{m+1}$. Note that $\underline{x}^2 = -|\underline{x}|^2 = -(x, x)$.

Recall that a continuously differentiable function f on an open domain $\mathcal{O} \subset \mathbb{R}^{m+1}$, with values on $\mathbb{C}_{m+1}$, is called (left) monogenic on $\mathcal{O}$ if it satisfies the Dirac equation (see, for example, [11,12,15])

$$\underline{D} f(x) = \sum_{j=1}^{m+1} e_j\, \partial_{x_j} f(x) = 0.$$

For $m = 1$, monogenic functions on $\mathbb{R}^2$ correspond to holomorphic functions of the complex variable $x_1 + e_1 e_2\, x_2$.

The Cauchy kernel,

$$E(x) = \frac{\overline{x}}{|x|^{m+1}},$$

is a monogenic function on $\mathbb{R}^{m+1} \setminus \{0\}$. In the spherical coordinates, $r = e^y = |\underline{x}|$, $\xi = \frac{\underline{x}}{|\underline{x}|}$, the Dirac operator reads

$$\underline{D} = \frac{1}{r}\, \underline{\xi} \left(r\partial_r + \Gamma_{\underline{\xi}} \right) = e^{-y} \underline{\xi} \left(\partial_y + \Gamma_{\underline{\xi}} \right), \tag{2.1}$$

where $\Gamma_{\underline{\xi}}$ is the spherical Dirac operator,

$$\Gamma_{\underline{\xi}} = -\underline{\xi}\, \partial_{\underline{\xi}} = -\sum_{i<j} e_{ij} \left(x_i \partial_{x_j} - x_j \partial_{x_i} \right).$$

We see from (2.1) that the equation for monogenic functions in the spherical coordinates is, on $\mathbb{R}^{m+1} \setminus \{0\}$, equivalent to

$$\underline{D}(f) = 0 \Leftrightarrow \partial_y f = -\Gamma_{\underline{\xi}}(f), \qquad r > 0. \tag{2.2}$$

The Laplacian Δ_x has the form

$$\Delta_x = \partial_r^2 + \frac{m}{r} \partial_r + \frac{1}{r^2} \Delta_{\underline{\xi}},$$

where $\Delta_{\underline{\xi}}$ is the Laplacian on the sphere (for the invariant metric). The relation between the spherical Dirac operator and the spherical Laplace operator is (see e.g. [12], (0.16) and section II.1)

$$\Delta_{\underline{\xi}} = \left((m-1)I - \Gamma_{\underline{\xi}} \right) \Gamma_{\underline{\xi}}. \tag{2.3}$$