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Modelling mean fields in networks of coupled oscillators

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ABSTRACT

We present the mathematical model of mean fields in complex networks of identical Kuramoto oscillators. In this framework, mean fields in the network are represented by the set of points in the unit disc with hyperbolic metric (Poincaré disc model). This set of points characterizes the network topology. The simulations for some random and regular graphs are presented.

1. Introduction

The classical Kuramoto model, studied in his seminal paper [1], describes globally (all-to-all) coupled heterogeneous population of phase oscillators:

$$\dot{\varphi}_j = \omega_j + \frac{K}{N}\sum_{i=1}^{N}\sin(\varphi_i - \varphi_j), \quad j = 1, \ldots, N. \tag{1}$$

Here, $\varphi_j(t)$ and ω_j denote the phase and the frequency of oscillator j, and K is a global coupling strength, the same for each pair of oscillators.

Probably the most important feature of this model is its *mean-field character*. In fact, Kuramoto started the analysis of (1) by introducing order parameter $r(t)$ and mean phase $\mu(t)$, defined by:

$$r(t)e^{i\mu(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{i\varphi_j(t)}. \tag{2}$$

In new variables, (1) can be rewritten in its *mean-field form*:

$$\dot{\varphi}_j = \omega_j + Kr\sin(\mu - \varphi_j), \quad j = 1, \ldots, N. \tag{3}$$

This form unveils the mean-field effect of the global coupling: it acts like all oscillators would be coupled to the same external field $\mu(t)$ with the common coupling strength $Kr(t)$ (see also [2]). This property makes the classical Kuramoto model so mathematically tractable. On the other hand, it is not trivial; its intrigue stems from the fact that both mean phase $\mu(t)$ and the strength of mean field $Kr(t)$ vary in time and depend on states of all oscillators at each instant of time.

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The same mean-field argument is valid for more general types of coupling, such as time-delayed coupling with global delay τ, phase-shifted coupling (Kuramoto–Sakaguchi model), time-dependent coupling $K(t)$, or noisy coupling with the common noise. It is essential that the coupling is global — that is, the same between each pair of oscillators.

It is natural to try the similar reasoning when dealing with the population of oscillators coupled through the complex network of interactions. Given the sufficiently large network, it can be conceived that contributions of all (or the greater part of) oscillators to collective dynamics are infinitesimally small (negligible) and interactions between oscillators can be approximated by introducing one or more mean fields. However, the situation is obviously far more complicated in this case, as the mean field does not have the same effect on all oscillators. More precisely, dynamics of different oscillators is governed by different mean fields. The challenge is to choose appropriate mathematical objects that describe mean fields in complex network.

In this paper, we introduce the mathematical model of mean fields in networks of coupled oscillators. Our approach, inspired by the paper [3], is a continuation and extension of [4] and can be used to characterize the network topology, detect communities, compare different networks, etc.

The concepts introduced in this paper are particularly transparent when applied to globally coupled population. In this case, there exists a global mean field that can be represented by a unique Möbius transformation at any moment. This is explained in Section 2. In Section 3 we discuss how this idea can be adapted when dealing with complex networks. In Section 4, the mathematical framework for description of mean fields in complex networks of oscillators is introduced. Based on it, we develop the statistical method of computation of mean fields and apply it to characterize some regular and random networks. Finally, in Section 5, we draw some conclusions and briefly discuss potential applications of our approach.

2. MMS principle and mean fields in globally coupled population

We start from the particularly simple case of the homogeneous population with global coupling, i.e. model (1) with all oscillators having the same intrinsic frequency $\omega_j \equiv \omega$. Following [3], rewrite (1) in more general form:

$$\dot{\varphi}_j = fe^{i\varphi_j} + \omega + \bar{f}e^{-i\varphi_j}, \quad j = 1, \dots, N, \tag{4}$$

where f is a global complex-valued *coupling function*. Introducing the new variable $z_j(t) = e^{i\varphi_j(t)}$, we represent the state of oscillator j as a point on the unit circle S^1 in the complex plane.

Denote by G the (sub)group of Möbius transformations that preserve the unit disc $\mathbb{D}$ in the complex plane. The general transformation belonging to G can be written as:

$$\mathcal{M}(z) = \frac{e^{i\psi}z + \alpha}{1 + \bar{\alpha}e^{i\psi}z}, \tag{5}$$

for some $\psi \in [0, 2\pi]$ and $\alpha \in \mathbb{C}$, $|\alpha| < 1$.

In [3], it is shown that the states $z_j(t)$ of globally coupled population of identical oscillators (4) evolve by the action of one-parametric family of transformations (5) with the parameters ψ and α satisfying the following system of ODEs [5]:

$$\begin{aligned} \dot{\alpha} &= i(f(t)\alpha^2 + \omega\alpha + \bar{f}(t)); \\ \dot{\psi} &= f(t)\alpha + \omega + \bar{f}(t)\bar{\alpha}. \end{aligned} \tag{6}$$

We will refer to this result as *MMS principle*.

Remark 1. MMS principle states that (4) admits many constants of motion and can be reduced to three-dimensional dynamics (6) of global variables $\alpha(t)$ and $\psi(t)$. It is important for our further considerations to emphasize one consequence that follows from the Lie group theory: given the initial states of oscillators $z_1(0), \dots, z_N(0)$, their states $z_1(t), \dots, z_N(t)$ at each moment t are obtained by the action of certain disc-preserving Möbius transformation. In other words, for each $t > 0$, one has $z_j(t) = \mathcal{M}_t(z_j(0))$, $j = 1, \dots, N$ for some $\mathcal{M}_t \in G$. Notice however, that it is impossible to specify this transformation $\mathcal{M}_t$ a priori, as it depends on coupling function f and states of all oscillators at each instant of time.

There are several possible ways to derive MMS principle. In [3] two methods are exposed, one based on analytic and another on algebraic and geometric arguments. Different method with more algebraic details is exposed in [6]. It is based on observation that the formal substitution $z_j(t) = e^{i\varphi_j(t)}$ in (4) yields complex Riccati ODEs:

$$\dot{z}_j = i(fz_j^2 + \omega z_j + \bar{f}). \tag{7}$$

First, notice that (7) with $\omega \in \mathbb{R}$ defines the *flow* on the unit circle — that is, given the initial condition $z_j(0)$ on S^1, one has that $z_j(t) \in S^1$ for any $t > 0$. Furthermore, it can be shown that Eq. (7) defines one-parametric family of disc-preserving Möbius transformations. Indeed, Poincaré maps of Riccati equations are Möbius transformations, see for instance [7,8].

However, there is an important nuance in the above reasoning. By referring to (7) as Riccati ODE, we implicitly assume that the coupling function f depends on t only, and not on $z_1, \dots, z_N$. This is essentially mean-field approximation; by adopting it we conceive that equations for the states of oscillators are coupled only through some common complex-valued function $f(t)$.