



Wallis formula from the harmonic oscillator

Ignacio Cortese, J. Antonio García *

Departamento de Física de Altas Energías, Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Apartado Postal 70-543, Ciudad de México, 04510, Mexico

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ABSTRACT

We show that the asymptotic formula for π , the Wallis formula, that was related with quantum mechanics and the hydrogen atom in Friedmann and Hagen (2015) can also be related to the harmonic oscillator using a quantum duality that relate these two systems. As a corollary we show that this very interesting asymptotic formula is not related with the hydrogen atom or quantum mechanics itself but with a clever choice of a trial function and a potential in the Schroedinger equation when we use the variational approach to calculate the ground state energy associated with the given potential function.

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1. Introduction

In the recent article [1] the authors claim that a venerable asymptotic formula for π , the Wallis (1655) product, is related to quantum mechanics. More precisely to the variational method applied to the ground state of the hydrogen atom in the limit of large angular momentum (the classical limit $\ell \rightarrow \infty$). The ingredients used are then a trial function for the variational method, the hamiltonian of the hydrogen atom and the classical limit. A recent analysis [2] show that if we change the trial function we can still obtain the Wallis formula or variants of it. The purpose of the present note is to show that the hydrogen atom hamiltonian is also not necessary to obtain the Wallis formula. We will show it explicitly using a well known duality between different potential functions in the Schroedinger equation. So perhaps there are other potentials and trial functions that can give rise to the Wallis asymptotic formula for π or variants of it. The classical limit that we will take is just the classical limit used in [1].

The Wallis product formula comes from a wise choice of the trial function in the variational approach and the potential function used in the Schroedinger equation. So nothing is magical here. It is interesting to see how some media reacted about this result and the cloud of misunderstanding that these texts induce [3]. Wallis formula comes from information that we put by hand (the trial function) and nothing reveals that this very nice product formula is inter constructed in quantum mechanics or the hydrogen atom themselves.

2. Duality in the Schroedinger equation

Consider the stationary Schroedinger equation in d spatial dimensions for a spherically symmetric potential $V(r) = K r^\beta$ with K and β some given constants. The radial part is

$$\left(-\frac{\hbar^2}{2m} \left(\frac{d^2}{dr^2} + \frac{(d-1)}{r} \frac{d}{dr} - \frac{\ell(\ell+d-2)}{r^2} \right) + V(r) - E \right) R(r) = 0. \quad (1)$$

* Corresponding author.

E-mail addresses: nachoc@nucleares.unam.mx (I. Cortese), garcia@nucleares.unam.mx (J.A. García).

By the change of variable

$$r = \rho^{2/(\beta+2)}, \tag{2}$$

the Schroedinger equation becomes

$$\left(-\frac{\hbar^2}{2m} \left(\frac{d^2}{d\rho^2} + \frac{(\beta + 2(d - 1))}{(\beta + 2)\rho} \frac{d}{d\rho} - \frac{\ell(\ell + d - 2)}{\rho^2} \left(\frac{2}{\beta + 2} \right)^2 \right) - \left(\frac{2}{\beta + 2} \right)^2 \rho^{-\frac{2\beta}{\beta+2}} E + \left(\frac{2}{\beta + 2} \right)^2 K \right) \tilde{R}(\rho) = 0, \tag{3}$$

where $\tilde{R}(\rho) \equiv R(r(\rho))$.

Now use the following dictionary (duality) to relate the relevant quantities E, K, d, ℓ that define the Schroedinger equation to a new set of quantities $\mathcal{E}, \mathcal{V}, D, \mathcal{L}$ that parametrize a new system:

• dimensions¹

$$D = \frac{2(\beta + d)}{\beta + 2} \tag{4}$$

• energy

$$\mathcal{E} = -\left(\frac{2}{\beta + 2} \right)^2 K \tag{5}$$

• angular momentum²

$$\mathcal{L} = \frac{2}{\beta + 2} \ell \tag{6}$$

• and the potential function

$$\mathcal{V} = -E \left(\frac{2}{\beta + 2} \right)^2 \rho^{-\frac{2\beta}{\beta+2}}. \tag{7}$$

Using this dictionary, Eq. (3) reads

$$\left(-\frac{\hbar^2}{2m} \left(\frac{d^2}{d\rho^2} + \frac{(D - 1)}{\rho} \frac{d}{d\rho} - \frac{\mathcal{L}(\mathcal{L} + D - 2)}{\rho^2} \right) + \mathcal{V}(\rho) - \mathcal{E} \right) \tilde{R}(\rho) = 0, \tag{8}$$

which has the *same* form as the Schroedinger equation for potential $\mathcal{V}(\rho)$ in D dimensions, with energy \mathcal{E} and angular momentum \mathcal{L} . This means that the map $r \rightarrow \rho$ and $R(r) \rightarrow \tilde{R}(\rho)$, together with (4)–(7), produces a solution for the problem with potential $\mathcal{V}(\rho)$ from a solution with potential $V(r)$.

Notice that we cannot map every solution of the stationary old problem into a stationary solution of the new problem by this duality. We can only map every *bounded* stationary solution of the old problem in a bounded stationary solution of a new problem.

We will use this dictionary for the case of the hydrogen atom, that is $\beta = -1$, in $d = 3$. In such case the new problem is the isotropic harmonic oscillator in $D = 4$ and angular momenta $\mathcal{L} = 2\ell$. This duality between the isotropic oscillator in $D = 4$ and the hydrogen atom in $d = 3$ was also presented in [5] where some details of this interesting relation at classical and quantum level not covered here are exposed with clear and deep insights.

3. Relation with the Wallis formula

The relation between the hydrogen atom and the Wallis formula is constructed from the trial function³

$$R(r) = e^{-ar^2} r^\ell. \tag{9}$$

The analysis given in [1] evaluates the minimum mean value of the hamiltonian in $d = 3$ with the trial function (9)

$$\langle H \rangle_{min}^\ell = -\frac{me^2}{2\hbar^2} \frac{1}{(\ell + \frac{3}{2})} \left(\frac{\Gamma(\ell + 1)}{\Gamma(\ell + \frac{3}{2})} \right)^2, \tag{10}$$

¹ The case $d = 2$ (conformal point) deserves special attention. See [4] for details.

² Here we are restricting ourselves to the case of integer new dimension D and integer new momenta \mathcal{L} .

³ We will not write the angular part because it will not play any role in the discussion.

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