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New non-naturally reductive Einstein metrics on exceptional simple Lie groups☆

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ABSTRACT

In this article, we construct several non-naturally reductive Einstein metrics on exceptional simple Lie groups, which are found through the decomposition arising from generalized Wallach spaces. Using the decomposition corresponding to the two involutions, we calculate the non-zero coefficients in the formulas of the components of Ricci tensor with respect to the given metrics. The Einstein metrics are obtained as solutions of a system of polynomial equations, which we manipulate by symbolic computations using Gröbner bases. In particular, we discuss the concrete numbers of non-naturally reductive Einstein metrics for each case up to isometry and homothety.

1. Introduction

A Riemannian manifold (M, g) is called Einstein if there exists a constant $\lambda \in \mathbb{R}$ such that $r = \lambda g$, where r is the Ricci tensor with respect to the metric g. We refer to Besse's book [1] for more details and results in this field before 1986. General existence results are difficult to obtain, and this fact leads many researchers to pay more attention on the special examples of Einstein manifolds. Among the first important attempts, the works of G. Jensen [2] and M. Wang, W. Ziller [3] made much contribution to the progress of this field. When the problem is restricted to Lie groups, D'Atri and Ziller in [4] obtained a large number of naturally reductive left-invariant metrics on compact simple Lie groups. Meanwhile, in the same paper they posed the problem of whether there exist non-naturally reductive Einstein metrics on a compact Lie group G.

In 1994, K. Mori [5] discovered the first left-invariant Einstein metrics on compact simple Lie groups $\mathrm{SU}(n)$ for $n \geq 6$, which are non-naturally reductive. In 2008, Arvanitoyeorgos, Mori and Sakane proved the existence of new non-naturally reductive Einstein metrics on $\mathrm{SO}(n)(n \geq 11)$, $\mathrm{Sp}(n)(n \geq 3)$, E_6, E_7 and E_8, using the method of fibrations on a compact simple Lie group over a Kähler C-space with two isotropy summands (see [6]). In 2014, using the methods of representation theory, Chen and Liang [7] found a non-naturally reductive Einstein metric on the compact simple Lie group F_4. More recently, Chrysikos and Sakane proved that there exists non-naturally reductive Einstein metric on exceptional Lie groups. In particular, they found the first example of non-naturally reductive Einstein metric on G_2 (see [8]). In 2017, Yan and Deng found a large number of non-naturally reductive Einstein metric on compact Lie groups from standard triples (see [9]). In [10], with a deep study of [11], H. Chen and Z. Chen achieved a lower boundary of the number of left-invariant non-naturally reductive Einstein metrics on compact Lie group $\mathrm{Sp}(n)$.

In this paper we consider new non-naturally reductive Einstein metrics on compact exceptional Lie groups G which can be viewed as principal bundles over generalized Wallach spaces $M = G/K$. In 2014, the classification of generalized Wallach

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spaces arising from a compact simple Lie group was obtained by Nikonorov [12] and Chen, Kang and Liang [13]. In particular, Nikonorov investigated the semisimple case and gave the classification in [12].

It is well known that the involutions play an important role in the development of homogeneous geometry. The Riemannian symmetric pairs were classified by Cartan [14], in the Lie algebra level, which can be treated as the structure of a Lie algebra with an involution satisfying some topological properties. Later on, the more general semisimple symmetric pairs were studied by Berger [15], whose classification can be obtained in the point view of involutions. Recently, Huang and Yu [16] classified the Klein four subgroups Γ of $\mathrm{Aut}(\mathfrak{u}_0)$ for compact Lie algebras $\mathfrak{u}_0$ up to conjugation by calculating the symmetric subgroups $\mathrm{Aut}(\mathfrak{u}_0)^\theta$ and their involution classes.

An interesting problem was raised by Y. Nikonorov in [17], which aims to finding an example of a compact Lie group G supplied with a left-invariant Riemannian ρ such that (G, ρ) is Einstein but is not a geodesic orbit Riemannian manifold. In that paper, by studying the results in [8] , Nikonorov found that there is 1 left-invariant non-naturally reductive Einstein metric on G_2 that is not geodesic. At the same time, he asked whether there is any other compact Lie group admitting such metrics. In our opinion, we think our results will make positive answers to this problem.

According to [13], each generalized Wallach space arising from a simple Lie group G is associated with two commutative involutions of $\mathfrak{g}$, the Lie algebra of G. With these two involutions, we have two different decompositions of $\mathfrak{g}$, which are in fact irreducible symmetric pairs. Based on the two irreducible symmetric pairs, we can get some linear equations for the non-zero coefficients in the expression of components of Ricci tensor with respect to the given metric. With the help of computer, we can find new Einstein metrics from the solutions of the system of polynomial equations.

The main results of this paper can be described as the following theorem.

Theorem 1.1.

(1) *The compact simple Lie group* F_4 *admits at least 3 new non-naturally reductive and non-isometric left-invariant Einstein metrics. These metrics are* $\mathrm{Ad}(\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SO}(5))$*-invariant.*
(2) *The compact simple Lie group* E_6 *admits at least 4 new non-naturally reductive and non-isometric left-invariant Einstein metrics. These metrics are* $\mathrm{Ad}(\mathrm{SU}(2) \times \mathrm{Sp}(3))$*-invariant.*
(3) *The compact simple Lie group* E_7 *admits at least 7 new non-naturally reductive and non-isometric left-invariant Einstein metrics. These metrics are* $\mathrm{Ad}(\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SO}(8))$*-invariant.*
(4) *The compact simple Lie group* E_8 *admits at least 13 new non-naturally reductive and non-isometric left-invariant Einstein metrics. Two of these metrics are* $\mathrm{Ad}(\mathrm{SO}(8) \times \mathrm{SO}(8))$*-invariant and the other 11 are* $\mathrm{Ad}(\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SO}(12))$*-invariant.*

The paper is organized as follows: In Section 2 we recall a formula for the Ricci tensor of G when we view G as a homogeneous space. In Section 3 we explain how to solve the non-zero coefficients in the expressions for Ricci tensor, where we will divide the exceptional Lie groups into several subclasses according to the numbers of simple ideals of $\mathfrak{k}$. Finally, in Section 4, for each case in Section 3, we discuss the non-naturally reductive Einstein metrics via the solutions of the systems polynomial equations, in particular, we will give the concrete numbers of non-naturally reductive Einstein metrics for each case.

2. The Ricci tensor of reductive homogeneous spaces

In this section we will recall a formula of the Ricci tensor with respect to a class of metrics on a compact semisimple Lie group and give a method to determine when a metric on G is naturally reductive.

Let G be a compact semisimple Lie group with Lie algebra $\mathfrak{g}$, and K a connected closed subgroup of G with Lie algebra $\mathfrak{k}$. Throughout this paper, we denote by B the negative of the Killing form of $\mathfrak{g}$, which is positive definite by the compactness of G. Then B can be viewed as an inner product on $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the orthogonal decomposition with respect to B such that $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$, where $\mathfrak{m}$ is the tangent space of G/K. In this paper, we assume that $\mathfrak{m}$ can be decomposed into mutually non-equivalent irreducible $\mathrm{Ad}(K)$-modules as follows:

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q.$$

Denote $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_p$, where $\mathfrak{k}_0 = Z(\mathfrak{k})$ is the center of $\mathfrak{k}$ and $\mathfrak{k}_i$ are simple ideals, $i = 1, \ldots, p$. Let $G \times K$ act on G by $(g_1, g_2)g = g_1 g g_2^{-1}$. Then $G \times K$ acts almost effectively on G with isotropy group $\Delta(K) = \{(k, k) | k \in K\}$. Hence G can be viewed as the coset space $(G \times K)/\Delta(K)$ and we have $\mathfrak{g} \oplus \mathfrak{k} = \Delta(\mathfrak{k}) \oplus \Omega$, where $\Omega \cong T_0((G \times K)/\Delta(K)) \cong \mathfrak{g}$ via the linear map $(X, Y) \to (X - Y) \in \mathfrak{g}, (X, Y) \in \Omega$.

It is well known that there exists an 1–1 correspondence between the G-invariant metrics on the reductive homogeneous space G/K and the $\mathrm{Ad}_G(K)$-invariant inner products on $\mathfrak{m}$. Let $(M = G/K, g)$ be a Riemannian reductive homogeneous space with $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K respectively, $\mathfrak{m}$ can be treated as the tangent space of M satisfying $\mathrm{Ad}(K)(\mathfrak{m}) \subset \mathfrak{m}$, then (M, g) is naturally reductive if

$$([X, Y]_{\mathfrak{m}}, Z) + (Y, [X, Z]_{\mathfrak{m}}) = 0, \quad \forall X, Y, Z \in \mathfrak{m},$$

where $(\ , \ )$ is the corresponding inner product on $\mathfrak{g}$.

In [4], D'Atri and Ziller studied the naturally reductive metrics among left-invariant metrics on compact Lie groups and obtained a complete classification of such metrics in the simple case. The following theorem will play an important role in deciding whether a left-invariant metric on a Lie group is naturally reductive.