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Approximate Noether symmetries and collineations for regular perturbative Lagrangians

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ABSTRACT

Regular perturbative Lagrangians that admit approximate Noether symmetries and approximate conservation laws are studied. Specifically, we investigate the connection between approximate Noether symmetries and collineations of the underlying manifold. In particular we determine the generic Noether symmetry conditions for the approximate point symmetries and we find that for a class of perturbed Lagrangians, Noether symmetries are related to the elements of the Homothetic algebra of the metric which is defined by the unperturbed Lagrangian. Moreover, we discuss how exact symmetries become approximate symmetries. Finally, some applications are presented.

1. Introduction

Symmetries play an important role in the study of differential equations. The existence of a symmetry vector implies the existence of a transformation which reduces the order of the differential equation (for ordinary differential equations) or the number of dependent variables (for partial differential equations). However, the existence of a symmetry vector indicates that there exists a curve in the phase space of the dynamical system which constrains the solution of the differential equation. This specific curve is a conservation law for the differential equation.

There are various ways to construct conservation laws, for instance see [1–4]. One of the most well-known, and simplest methods for the determination of conservation laws is the application of Noether's theorems [5]. In particular, the first Noether's theorem states that, if the Lagrangian function which describes a dynamical system changes under the action of a point transformation such that the Action integral is invariant, the dynamical system is also invariant under the action of the same point transformation. Moreover, a conservation law corresponds to this point transformation according to Noether's second theorem.

Usually, when we refer to symmetries, we consider the exact symmetries. However, for perturbative dynamical systems the context of symmetries is extended and the so-called approximate symmetries are defined [6–13]. In this work we are interested in the application of Noether's theorem for approximate symmetries on some regular perturbative Lagrangians.

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Approximate Noether symmetries [14,15] provide approximate first integrals, functions which can be used as conservation laws until a specific step in numerical integrations. This kind of approximate conservation laws have played an important role for the study of chaotic systems — for an extended discussion we refer the reader to an application in Galactic dynamics [16–18].

Whilst recently, the advent of automated software algorithms has made light work of calculating symmetries [19]. Such programs are often limited by models involving many variables or higher-order perturbations. This problem, in part, has fueled the need to write this paper. Here, we take a compound problem, whereby scientists have previously relied on numerical techniques for analysis, and instead frame it in the context of an analytical scheme. We present a set of conditions that may be specialized for appropriate Lagrangian functions that necessarily contains a perturbation. Inspired by the approach of Tsamparlis and Paliathanasis [20–22], we show how those conditions can be solved with the use of some theorems from differential geometry. Indeed, geometric based theories have far reaching applications [23–25].

Specifically, in this paper, the Noether conditions, or symmetry determining system of equations, are formulated by contemplating point transformations in ascending order of the perturbation parameter ε. To illustrate the advantages of such a formulation, the general conditions are applied to the perturbations of oscillator type equations corresponding to n dimensions. Moreover, we discuss the admitted approximate conserved quantities for symmetries of first-order up-to nth-order.

This paper assumes familiarity with symmetry-based methods and so does not add to the volume of the work by recapitulating the well established theory. However, in stating this we recommend the interested reader to consult the books [26] and [27], which contain various aspects concerning Lie and Noether symmetries.

The objective of this paper is two-fold. First, we utilize a generalized Lagrangian to formulate Noether symmetry conditions, for symmetries of higher-order perturbations. Thereafter, the latter is used to establish the corresponding approximate conservation laws, also of higher-order perturbations, via Noether's theorem. In this regard, to cope with the complexity of our derivation, we employ the Einstein summation convention, and require that indices enclosed in parentheses indicate symmetrization, for instance, $F_{(ij)} = \frac{1}{2}(F_{ij} + F_{ji})$.

The family of perturbed Lagrangians that we assume are

$$L\left(t, x^k, \dot{x}^k, \varepsilon\right) = L_0\left(t, x^k, \dot{x}^k\right) + \varepsilon L_1\left(t, x^k, \dot{x}^k\right) + O\left(\varepsilon^2\right), \tag{1}$$

where we stipulate that the exact and approximate terms are defined by the regular Lagrangians

$$L_0\left(t, x^k, \dot{x}^k\right) = \frac{1}{2} g_{ij}\dot{x}^i\dot{x}^j - V_0\left(t, x^k\right), \tag{2}$$

$$L_1\left(t, x^k, \dot{x}^k\right) = \frac{1}{2} h_{ij}\dot{x}^i\dot{x}^j - V_1\left(t, x^k\right), \tag{3}$$

respectively, where $g_{ij} = g_{ij}\left(x^k\right), h_{ij} = h_{ij}\left(x^k\right)$ and dot denotes total derivative with respect to the independent parameter t, i.e. $\dot{x}^i = \frac{dx^i}{dt}$. Such Lagrangians are not limited by any specific assumptions in order to preserve generality. The plan of the paper is as follows.

Section 2 is the main core of our analysis where we derive the approximate Noether symmetry conditions for regular Lagrangians of order $\left(\varepsilon^1\right)$ and (ε^n) in general. Moreover, from the conditions we see that there exists a link between the approximate symmetries and the Homothetic vector fields of the metric tensor g_{ij}. In Section 3 we demonstrate our results by applying our approach to various examples of maximally symmetric systems, and also on a $sl\,(2, R)$ exact invariant system. For each case we derive the approximate Noether symmetries and the corresponding approximate conservation laws. Finally, in Section 4 we discuss our results and draw our conclusions.

2. Approximate Noether conditions

In the interest of clarity and completeness we have decided to present the immediate work that follows in great detail, dividing the procedure into important steps. Following Govinder et al. [14], under a point transformation

$$\bar{x}^i = x^i + a\left(\eta^i_{(0)} + \varepsilon\eta^i_{(1)} + O\left(\varepsilon^2\right)\right), \tag{4}$$

$$\bar{t} = t + a\left(\xi_{(0)} + \varepsilon\xi_{(1)} + O\left(\varepsilon^2\right)\right), \tag{5}$$

with $\xi_A = \xi_A\left(t, x^k\right)$, $\eta^i_A = \eta^i_A\left(t, x^k\right)$, $A = 0, 1$ and "a, ε" as two infinitesimal parameters, we have the generator

$$X = X_0 + \varepsilon X_1 + O\left(\varepsilon^2\right), \tag{6}$$

where $X_A = \xi_A\partial_t + \eta^i_A\partial_i$. Here, X is a first-order approximate vector field composed of an exact (X_0) and an approximate (X_1) part. The generator (6) is a Noether point symmetry if the following equation is satisfied

$$X^{[1]}L + L\frac{d\xi}{dt} = \frac{df}{dt}, \tag{7}$$