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On quasi-symplectic groupoid reduction

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ABSTRACT

Hamiltonian systems on quasi-symplectic groupoid reduced spaces are studied. Moreover, for product quasi-symplectic groupoids, we consider Hamiltonian reduction by stages and prove a commutating reduction theorem.

1. Introduction and main results

Quasi-symplectic groupoids and their moment map theory are introduced by Xu in [1] to unify into a single framework various reduction theories such as Marsden–Weinstein quotients [2], reduction by symplectic groupoids [3], and reduction by Lie group valued moment maps [4].

A quasi-symplectic groupoid $(\mathcal{G} : G \rightrightarrows M, \omega + \Omega)$ consists of a Lie groupoid $\mathcal{G} : G \rightrightarrows M$ and two differential forms $\omega \in \Omega^2(M)$ and $\Omega \in \Omega^3(G)$, in which ω satisfies a weak non-degeneracy condition, and $\omega + \Omega$ is a cocycle in the de Rham complex of the Lie groupoid $\mathcal{G} : G \rightrightarrows M$.

Given a quasi-symplectic groupoid $(\mathcal{G} : G \rightrightarrows M, \omega + \Omega)$ and a Hamiltonian $\mathcal{G}$-space $(X, \omega_X, J : X \to M)$. Under suitable conditions, for example, if $m \in M$ is a regular value of J and the isotropic subgroup $G(m)$ acts on $J^{-1}(m)$ freely and properly, then the quotient space $J^{-1}(m)/G(m)$ is a smooth manifold endowed naturally with a symplectic form. Moreover, if $(\mathcal{G}_i : G_i \rightrightarrows M_i, \omega_i + \Omega_i)$ are two quasi-symplectic groupoids such that $(X, \omega_X, J : X \to M_1 \times M_2)$ is a Hamiltonian $(\mathcal{G}_1 \times \mathcal{G}_2)$-space, then the quotient space $\overline{X}_{m_2} := J_2^{-1}(m_2)/G_2(m_2)$ (if is a smooth manifold) is a Hamiltonian $\mathcal{G}_1$-space. So we can perform the reduction process again and get a new reduced space $(\overline{X}_{m_2})_{m_1}$. It is natural to compare this space to $\overline{X}_{(m_1,m_2)}$ which is reduced by X under the product groupoid $\mathcal{G}_1 \times \mathcal{G}_2$. We will show that these two spaces are symplectically homeomorphic. This result is a groupoid version of "Hamiltonian reduction by stages" [5].

We list here the main results of this paper:

(1) In Theorem 3.3, for each $\mathcal{G}$-invariant smooth function $F : X \to \mathbb{R}$, we define on X a unique $\mathcal{G}$-invariant vector field X_F which preserves ω_X and $J : X \to M$. By using these vector fields, a Poisson structure can be defined on the space $C^\infty(X; \mathbb{R})^{\mathcal{G}}$ of $\mathcal{G}$-invariant smooth functions on X.
(2) In Theorem 3.6, we construct the reduced Hamiltonian systems on reduced spaces.

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(3) The product $\mathcal{G}_1 \times \mathcal{G}_2$ of two quasi-symplectic groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ is naturally a quasi-symplectic groupoid. Suppose $(X, \omega_X, J_i : X \to M_i)$ is a Hamiltonian $\mathcal{G}_i$-space for each $i = 1, 2$. Under suitable conditions, in Theorem 4.2, we prove $(X, \omega_X, J_1 \times J_2)$ is a Hamiltonian $\mathcal{G}_1 \times \mathcal{G}_2$-space.

(4) In Theorem 5.4, we prove a commutating reduction theorem for quasi-symplectic groupoid reduction theory. We show that under a sufficient condition, the reduced space of (X, ω_X) by the product $\mathcal{G}_1 \times \mathcal{G}_2$ is symplectically homeomorphic to the reduced space of (X, ω_X) by $\mathcal{G}_1$ and $\mathcal{G}_2$ in two stages.

2. Preliminaries

In this section we recall the definition and some basic properties of quasi-symplectic groupoids. All of them can be found in [1].

Let $\mathcal{G} : G \rightrightarrows M$ be a Lie groupoid with source map $s : G \to M$, target map $t : G \to M$, multiplication map $m : G{\times}_M G \to G, (g, h) \mapsto gh$ if $g, h \in G$ with $t(g) = s(h)$, and unit map $\epsilon : M \to G$. See [6] and [7] for the definition and basic properties of Lie groupoids.

Definition 2.1. A Lie groupoid $\mathcal{G} : G \rightrightarrows M$ is called a *quasi-symplectic groupoid* if there exists a 3-form Ω on M and a 2-form ω on G such that

(Q1) $d\Omega = 0, d\omega = s^*\Omega - t^*\Omega, pr_1^*\omega + pr_2^*\omega = m^*\omega$. Here, $pr_i : G{\times}_M G \to G$ are the natural projections for $i = 1, 2$;

(Q2) for every $m \in M$, the anchor map

$$ds(m) : \ \ker \omega(m) \cap T_m t^{-1}(m) \to \ker \omega(m) \cap T_m M$$

is an isomorphism. Here we identify M with its image $\epsilon(M)$ in G.

Remark 2.2. The second condition that each anchor map $ds(m) : \ \ker \omega(m) \cap T_m t^{-1}(m) \to \ker \omega(m) \cap T_m M$ is an isomorphism is equivalent to ([1, Proposition 2.7])

each such anchor map is injective, and $\dim G = 2 \dim M$,

and hence is also equivalent to

(Q2′) $\ker ds(m) \cap \ker dt(m) \cap \ker \omega(m) = 0$, and $\dim G = 2 \dim M$.

Bursztyn–Crainic–Weinstein–Zhu [8] use conditions (Q1) and (Q2′) to define the same groupoids under the name *twisted presymplectic groupoids*.

Let us recall some properties of quasi-symplectic groupoids:

Lemma 2.3. *Let $(\mathcal{G} : G \rightrightarrows M, \omega + \Omega)$ be a quasi-symplectic groupoid. Then*

(i) *the 2-form ω vanishes on M, that is, $\epsilon^*\omega = 0$;*

(ii) *for all $m \in M$, we have the following decompositions*

$$T_m G = T_m t^{-1}(m) \oplus T_m M = T_m s^{-1}(m) \oplus T_m M.$$

Proof. The first one is (1) in [1, Proposition 2.3]. The last one is because every tangent vector $\zeta \in T_m G$ can be written as a sum

$$\zeta = (\zeta - dt(m)\zeta) + dt(m)\zeta \in T_m t^{-1}(m) + T_m M.$$

The decomposition is clearly a direct sum. □

Definition 2.4. Let $\mathcal{G} : G \rightrightarrows M$ be a Lie groupoid and X a smooth manifold. We say $\mathcal{G}$ *acts on X from the left*, or X is a *left $\mathcal{G}$-space*, if there exist smooth maps $J : X \to M$, and $G{\times}_M X \to X, (g, x) \mapsto g \cdot x$, such that for every $(g, x) \in G{\times}_M X$ and every $h \in G$ with $t(h) = s(g)$,

$$J(g \cdot x) = s(g), \ \ h(g \cdot x) = (hg) \cdot x, \ \ \epsilon(J(x)) \cdot x = x.$$

Here, $G{\times}_M X = \{(g, x) \mid t(g) = J(x)\}$ is a smooth submanifold of $G \times X$. The smooth map $J : X \to M$ is called the *moment* map.

Remark 2.5. It is well-known (see, e.g. [7, p. 125], in different conventions) that given a left $\mathcal{G}$-space $(X, J : X \to M)$, we have the translation groupoid $G{\times}_M X \rightrightarrows X$ whose source map $s_X : G{\times}_M X \to X$ is the action $(g, x) \mapsto g \cdot x$ and whose target map $t_X : G{\times}_M X \to X$ is the natural projection $(g, x) \mapsto x$.

The action of $\mathcal{G}$ on X induces an infinitesimal action: Given a point x in X. Write $m = J(x) \in M$. Then the $\mathcal{G}$ action induces a smooth map

$$t^{-1}(m) \to X; \quad g \mapsto g \cdot x,$$