Journal of Geometry and Physics 124 (2018) 371–412

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Harmonic maps from super Riemann surfaces

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ARTICLE INFO

Article history:
Received 31 March 2017
Accepted 28 October 2017
Available online 11 November 2017

Keywords:
Harmonic maps
Super Riemann surfaces
Finite type
Completely integrable systems

ABSTRACT

In this paper we study harmonic maps from super Riemann surfaces in complex projective spaces and projective spaces associated with the super skew-field $\mathbb{D}$. In both cases, we develop the theory of Gauß transforms and study the notion of isotropy, in particular its relation to holomorphic differentials on the super Riemann surface.

Moreover, we give a definition of finite type harmonic maps for a special class of maps into $\mathbb{C}P^{n|n+1}$ and thus obtain a classification for certain harmonic super tori. Furthermore, we investigate the equations satisfied by the underlying objects and give an example of a harmonic super torus in $\mathbb{D}P^2$ whose underlying map is not harmonic.

1. Introduction

The purpose of this article is to prove some foundational results about harmonic maps in supergeometry. More precisely, we study harmonic maps from a super Riemann surface into complex projective spaces and in special cases into general complex Grassmannians.

Harmonic maps from Riemann surfaces into various target spaces are by now a classical topic in differential geometry. Such maps occur naturally in surface theory, for instance. The parametrization of a surface in $\mathbb{R}^3$ is minimal if and only if it is conformal and harmonic. It has constant mean curvature if and only if its Gauß map is harmonic. We refer to [1] for a treatment of these results. In the context of the anti-self-dual Yang–Mills equation, such maps appear as a symmetry reduction from four to two dimensions [2]. Consequently, a central problem is to develop techniques which allow for a classification and construction of such maps. For a review of this broad subject, we refer the reader to the survey articles [3,4]. Closer to the specific subject of the present article are [5–8].

Supergeometry is the extension of ordinary geometry which allows for commuting and anti-commuting coordinate functions. Many notions, constructions, and results from differential geometry carry over to the graded setting directly. In particular, there is a notion of Riemannian supermanifolds. However, the Riemannian structure might be even or odd. Another genuinely supergeometric notion is supersymmetry, the simplest instance of which is the concept of a super Riemann surface. The complex analytic properties have been studied in pioneering works in the 1980s, among others [9,10], and more recently in [11,12].

In this setting there exists a natural notion of harmonic maps from super Riemann surfaces to Riemannian supermanifolds [13–16] which are the central objects of this article. It is beyond the scope, to give a comprehensive treatment, rather we will concentrate on some selected aspects.

In order to put our results into context, we first give a brief account on the relevant results in the ungraded setting. The energy of a map $f : \Sigma \to M$ between a compact Riemann surface and a Riemannian manifold is defined by

$$E(f) = \int_\Sigma \langle df_{\mathbb{C}}|_{T\Sigma^{(0,1)}}, df_{\mathbb{C}}|_{T\Sigma^{(1,0)}} \rangle_{\mathbb{C}},$$

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https://doi.org/10.1016/j.geomphys.2017.10.017
0393-0440/

where $\langle -, -\rangle_{\mathbb{C}}$ denotes the complex bilinear extension of the given Riemannian structure to $TM_{\mathbb{C}}$. Critical points are called harmonic maps and are characterized in a local complex coordinate z by

$$\nabla^{LC}_{\partial_{\bar{z}}}(df_{\mathbb{C}})(\partial_z) = 0,$$

where ∇^{LC} denotes the pullback of the Levi-Civita connection. This connection on $f^*(TM_{\mathbb{C}})$ gives rise to the Koszul–Malgrange holomorphic structure and, using this, f is harmonic if and only if $(df_{\mathbb{C}})|_{T\Sigma^{(1,0)}}$ is a holomorphic section of $(T\Sigma^{(1,0)})^* \otimes TM_{\mathbb{C}}$. In particular, the differential either vanishes identically or its zeros are isolated.

In the case $M = \mathbb{C}P^n$, the harmonic map equation is equivalent to

$$\nabla^{LC}_{\partial_{\bar{z}}} df^{(1,0)}(\partial_z) = 0, \tag{1.1}$$

where $df_{\mathbb{C}} = df^{(1,0)} + df^{(0,1)}$ according to the type decomposition on $\mathbb{C}P^n$. In view of the isomorphism $(T\mathbb{C}P^n)^{(1,0)} \cong \mathrm{Hom}(\gamma, \gamma^{\perp})$, where γ is the tautological line bundle, if f is not antiholomorphic, $df^{(1,0)}(\partial_z)$ defines a line in $\mathbb{C}^{1+n}$ outside a discrete set of points. One can always extend this to a give a new map $f_1 : \Sigma \to \mathbb{C}P^n$, the Gauß transform. If f is not holomorphic, one can similarly produce a new map f_{-1} starting from $df^{(1,0)}(\partial_{\bar{z}})$. The central observation is that $f_{\pm 1}$ are harmonic again [17,18]. This process can be iterated and gives the harmonic sequence

$$\ldots, f_{-2}, f_{-1}, f, f_1, f_2, \ldots$$

The harmonic map is called isotropic if this sequence is finite

$$f_{-l(f)}, \ldots, f_{-2}, f_{-1}, f, f_1, f_2, \ldots, f_{k(f)},$$

which forces the leftmost (resp. rightmost) map to be holomorphic (resp. antiholomorphic). Such can thus be described by means of complex geometry (cf. [18, Thm. 6.9]). For a Riemann sphere, any harmonic map is isotropic, so that this theorem accounts for all full harmonic maps.

However, this is not necessarily the case for a torus $T^2 = \mathbb{C}/\Omega$. We shall especially be interested in the case where the map is $(n+1)$-orthogonal, meaning that any consecutive $n+1$ lines in the harmonic sequence are mutually orthogonal, and non-isotropic. The harmonic sequence is in this situation infinite and in fact periodic, $f_k = f_{n+1+k}$.

Remark 1.2. In [19], harmonic maps of this type are called superconformal. In view of the next section and the following material, this terminology would be very unfavourable in the context of the present article.

The classification result for such maps is quite different in nature compared to the previous result and is based on the notion of harmonic maps of finite type. This approach has been developed and applied in a series of papers [20–23]. The special situation we consider was dealt with in the ungraded case in [19]. The case of general harmonic tori in $\mathbb{C}P^n$ has been settled in [24].

In order to explain this notion and the results, we need to back up and introduce new objects. In the case at hand, the harmonic sequence determines a lift

$$\tilde{f} : \Sigma \to SU(n+1)/T,$$

where T is a maximal torus. The relevant structure of the co-domain is the structure of a $(n+1)$-symmetric space, i.e., it is equipped with an automorphism of order $n+1$, which leads after complexification to a decomposition

$$\mathfrak{psl}(n+1) = \bigoplus_{i=0}^{n} \mathcal{M}_i.$$

At this point, the only special property of these eigenspaces is the following. The pullback of $\tilde{f}$ along $p : \mathbb{C} \to T^2$ has a lift $F : \mathbb{C} \to SU(n+1)$ and it follows from the definition of the Gauß transform, that the pullback of the Maurer–Cartan form along F takes the form

$$F^*\alpha_z = A_{z,0} + A_{z,1}, \tag{1.3}$$

where $A_{z,i}$ takes values in $\mathcal{M}_i$ and $A_{z,1}$ satisfies a non-degeneracy condition given in terms of an invariant polynomial. This is actually a property of the map $\tilde{f}$ and such maps are called primitive. The concept of finite type harmonic maps is to construct solutions to (1.3) by solving two commuting ordinary differential equations. Then any of $f, \tilde{f}$, or F is called of finite type if it can be obtained from this construction. (This will be made more precise in our situation in Section 6.4.4.)

These commuting ordinary differential equations are constructed from the real and imaginary part of a complex vector field defined on

$$\Lambda_d = \{\sum_{i=-d}^{d} \xi_i \lambda^i \mid \xi \in \mathfrak{psl}(n+1),\ \bar{\xi}_i = \xi_{-i}\},\ \ \Lambda_{d,\tau} = \{\xi \in \Lambda_d \mid \xi_i \in \mathcal{M}_i\},$$

where $d \equiv 1 \pmod{n+1}$. This is given by

$$Z(\xi) = [\xi, \frac{1}{2}\xi_{d-1} + \lambda \xi_d] \tag{1.4}$$