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Differential invariants for spherical flows of a viscid fluid

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ABSTRACT

Symmetries and the corresponding algebras of differential invariants of viscid fluids on a sphere are given. Their dependence on thermodynamical states of media is studied, and a classification of thermodynamical states is given.

1. Introduction

In this paper we study differential invariants of flows of compressible viscid fluids or gases on a sphere.

The system of differential equations (the Navier–Stokes system) governing flows on an oriented Riemannian manifold (M, g) consists of the following equations:

$$\begin{cases} \rho(\mathbf{u}_t + \nabla_{\mathbf{u}}\mathbf{u}) - \operatorname{div}\sigma = 0, \\ \dfrac{\partial(\rho\,\Omega_g)}{\partial t} + \mathcal{L}_{\mathbf{u}}\left(\rho\,\Omega_g\right) = 0, \\ \rho T\,(s_t + \nabla_{\mathbf{u}} s) - \Phi - k(-\Delta_g T) = 0, \end{cases} \tag{1}$$

where the vector field $\mathbf{u} = (u, v)$ is the flow velocity, p, ρ, s, T are the pressure, density, entropy, temperature of the fluid respectively, and k is the thermal conductivity, which is supposed to be constant.

Here ∇_X is the directional covariant Levi-Civita derivative with respect to a vector field X, Ω_g is the volume form on the manifold M, Δ_g is the Beltrami–Laplace operator corresponding to the metric g.

The divergence operator $\operatorname{div} : S^2T^*M \to TM$ is given by

$$\operatorname{div}\sigma = (d_\nabla \sigma)_{ijk} g^{jk} g^{il},$$

where d_∇ is the covariant differential.

The fluid stress tensor σ is supposed to be linearly dependent on the rate of deformation tensor $D = \frac{1}{2}\mathcal{L}_{\mathbf{u}}(g)$.

Since the medium is isotropic the stress tensor $\sigma = -pg + \sigma'$ is symmetric. The viscous stress tensor σ' has the form

$$\sigma' = 2\eta\left(D - \frac{\langle D, g\rangle_g}{\langle g, g\rangle_g} g\right) + \zeta\,\langle D, g\rangle_g g,$$

which is analogous to the case of plane flows.

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Quantity $\Phi = \langle \sigma', D \rangle_g$ is the rate of dissipation of mechanical energy [1].

We consider the case of a two-dimensional unit sphere $M = S^2$ with the metric $g = \sin^2 y\, dx^2 + dy^2$ in the spherical coordinates.

The first equation of system (1) is the 2-dimensional Navier–Stokes equation, the second is the expression of the conservation mass law and the third is the general equation of heat transfer. See also [1] for details.

Since system (1) has four equations for the six unknown functions it is incomplete. We obtain additional equations using the thermodynamics of the medium.

The paper is organized as follows. In Section 2 we recall, compare with [2], main thermodynamic notions in terms of Lagrangian surfaces.

In Section 3 we discuss symmetry Lie algebras of the Navier–Stokes system on the sphere and their dependence on the thermodynamic state. The symmetry algebra consists of pure geometrical and thermodynamic parts. The geometrical part consists of sphere motions and time shifts. The thermodynamic part depends on the symmetries of the thermodynamic state.

In Section 4 we consider the thermodynamic states and the corresponding Lie algebras for the cases when the thermodynamic state admits a one-dimensional symmetry algebra.

The symmetry algebra that is realized for general thermodynamic states depends only on the geometry of the medium. We call the corresponding differential invariants *kinematic.*

In Section 5 a full description of the algebra of kinematic invariants is given. Depending on a symmetry of the thermodynamic states we get a bigger symmetry algebra and therefore a smaller algebra of invariants (we call them Navier–Stokes invariants). A description of these algebras for the case of the thermodynamic states with one-dimensional symmetry algebras is given. The Navier–Stokes invariants give the complete information about the flow as well as the medium.

Many of the computations in this paper were done in Maple with the Differential Geometry package [3] by I. Anderson and his team. Maple files with the most important computations in this paper can be found on the web-site http://d-omega.org/appendices/.

2. Thermodynamics

For the further discussion we need the thermodynamical principles expressed in geometrical form, which are found in [4]. Here we only briefly recall them.

To complete PDE system (1) we need two additional relations on the thermodynamic quantities used in the system.

We have the following thermodynamic quantities: the specific volume ρ^{-1}, the specific entropy s and the specific internal energy ϵ.

Let us consider a 5-dimensional contact manifold $\Phi = \mathbb{R}^5$ equipped with coordinates $(p, \rho, s, T, \epsilon)$ and the contact 1-form

$$\theta = d\epsilon - T ds - \frac{p}{\rho^2} d\rho.$$

Then the thermodynamical states are a two-dimensional Legendrian manifold L, i.e. such surface $L \subset \Phi$, that the first law of thermodynamics $\theta\big|_L = 0$ holds.

We will consider the case, when the functions (ρ, s) are coordinates on the manifold L. Then this surface can be defined by the structure equations:

$$\epsilon = \epsilon(\rho, s), \quad T = \frac{\partial \epsilon}{\partial s}, \quad p = \rho^2 \frac{\partial \epsilon}{\partial \rho}.$$

Since the Navier–Stokes system does not depend on the specific internal energy ϵ it must be eliminated from the description of the thermodynamic states. Consider the projection $\phi : \mathbb{R}^5 \to \mathbb{R}^4$, $\phi : (p, \rho, s, T, \epsilon) \longmapsto (p, \rho, s, T)$. The restriction of the map ϕ on the state surface L is a diffeomorphism on the image $\bar{L} = \phi(L)$ and the surface $\bar{L} \subset \mathbb{R}^4$ is a Lagrangian manifold in the 4-dimensional symplectic space $\mathbb{R}^4$ equipped with the structure form

$$\Omega = ds \wedge dT + \rho^{-2} d\rho \wedge dp.$$

Therefore, equivalently, the thermodynamic states can be considered as the Lagrangian submanifolds in the symplectic space $(\mathbb{R}^4, \Omega)$.

Thus, if we define the two-dimensional surface $\bar{L}$ by the equations

$$\begin{cases} F(p, \rho, s, T) = 0, \\ G(p, \rho, s, T) = 0, \end{cases} \tag{2}$$

then the condition for the surface $\bar{L}$ to be Lagrangian is formulated as follows:

$$[F, G] = 0 \text{ on } \bar{L}, \tag{3}$$

where $[F, G]$ is the Poisson bracket with respect to the symplectic form Ω.