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S-Spectrum and the quaternionic Cayley transform of an operator

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ABSTRACT

In this paper we define the quaternionic Cayley transformation of a densely defined, symmetric, quaternionic right linear operator and formulate a general theory of defect number in a right quaternionic Hilbert space. This study investigates the relation between the defect number and S-spectrum, and the properties of the Cayley transform in the quaternionic setting.

1. Introduction

Self-adjoint operators play an important role in the Dirac–von Neumann formulation of quantum mechanics. In complex and in quaternionic quantum mechanics states are described by vectors of a separable complex (resp. quaternionic) Hilbert space and the observables are represented by self-adjoint operators on the respective Hilbert space. By Stone's theorem on one parameter unitary groups, self-adjoint operators are the infinitesimal generators of unitary groups of time evolution.

The self-adjointness in a Hilbert space is stronger than being symmetric. Even though the difference is a technical issue, it is very important. For example, the spectral theorem only applies to self-adjoint operators but not to symmetric operators. In this regard, the following question arises in several contexts: if an operator A on a Hilbert space is symmetric, when does it have self-adjoint extensions? In the complex case, an answer is provided by the Cayley transform of a self-adjoint operator and the deficiency indices.

Due to the non-commutativity, in the quaternionic case there are three types of Hilbert spaces: left, right, and two-sided, depending on how vectors are multiplied by scalars. This fact can entail several problems. For example, when a Hilbert space $\mathcal{H}$ is one-sided (either left or right) the set of linear operators acting on it does not have a linear structure. Moreover, in a one sided quaternionic Hilbert space, given a linear operator T and a quaternion $\mathfrak{q} \in \mathbb{H}$, in general we have that $(\mathfrak{q}T)^{\dagger} \neq \overline{\mathfrak{q}}T^{\dagger}$

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(see [1] for details). These restrictions can severely prevent the generalization to the quaternionic case of results valid in the complex setting. Even though most of the linear spaces are one-sided, it is possible to introduce a notion of multiplication on both sides by fixing an arbitrary Hilbert basis of $\mathcal{H}$. This fact allows to have a linear structure on the set of linear operators, which is a minimal requirement to develop a full theory. Thus, the framework of this paper is a right quaternionic Hilbert space equipped with a left multiplication, introduced by fixing a Hilbert basis. As in the complex case, one may introduce a suitable notion of Cayley type transform of symmetric linear operators. The idea of considering Cayley transform of linear operators is due to von Neumann [2] who formally replaced the variable in a Cayley transform by a symmetric operator. The idea was further extended to other types of linear operators but always with the purpose of getting information of the given operator by studying the properties of its Cayley transform. A quaternionic Cayley transform of linear operators appeared in [3,4]; however, the type of transform and the underlying notion of spectrum differ from the one treated in this paper.

In this paper, we define the Cayley transform of densely defined symmetric operators satisfying suitable assumptions. We will prove that this notion of Cayley transform possesses several properties, in particular it is an isometry and allows to prove a characterization of self-adjointness.

The plan of the paper is the following. The paper consists of four sections, besides the Introduction. In Section 2 we collect some preliminary notations and results on quaternions, quaternionic Hilbert spaces and Hilbert bases. In Section 3 we introduce right linear operators and some of their properties, the left multiplication, we introduce the notion of deficiency subspace and defect number of an operator at a point also proving some new results in this framework. In Section 4 we study the deficiency indices of isometric operators and we define the notion of quaternionic Cayley transform for a linear symmetric operator (satisfying suitable hypotheses) and study its main properties. In particular, we show that a linear operator is self-adjoint if and only if its Cayley transform is unitary. In the fifth and last section, we show that the Cayley transform that we have defined based on the choice of a Hilbert basis, in order to have a left multiplication and thus a two-sided Hilbert space, in fact does not depend on this choice.

2. Mathematical preliminaries

In order to make the paper self-contained, we recall some facts about quaternions which may not be well-known. For details we refer the reader to [5–7].

2.1. Quaternions

Let $\mathbb{H}$ denote the field of all quaternions and $\mathbb{H}^*$ the group (under quaternionic multiplication) of all invertible quaternions. A general quaternion can be written as

$$\mathfrak{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}, \qquad q_0, q_1, q_2, q_3 \in \mathbb{R},$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the three quaternionic imaginary units, satisfying $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$, $\mathbf{jk} = \mathbf{i} = -\mathbf{kj}$, $\mathbf{ki} = \mathbf{j} = -\mathbf{ik}$. The quaternionic conjugate of $\mathfrak{q}$ is

$$\overline{\mathfrak{q}} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3,$$

while $|\mathfrak{q}| = (\mathfrak{q}\overline{\mathfrak{q}})^{1/2}$ denotes the usual norm of the quaternion $\mathfrak{q}$. If $\mathfrak{q}$ is non-zero element, it has inverse $\mathfrak{q}^{-1} = \frac{\overline{\mathfrak{q}}}{|\mathfrak{q}|^2}$. Finally, the set

$$\mathbb{S} = \{I = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \mid x_1, x_2, x_3 \in \mathbb{R},\ x_1^2 + x_2^2 + x_3^2 = 1\},$$

contains all the elements whose square is -1. It is a 2-dimensional sphere in $\mathbb{H}$ identified with $\mathbb{R}^4$.

2.2. Quaternionic Hilbert spaces

In this subsection we discuss right quaternionic Hilbert spaces. For more details we refer the reader to [5–7].

2.2.1. Right quaternionic Hilbert Space

Let $V_{\mathbb{H}}^R$ be a vector space under right multiplication by quaternions. For $\phi, \psi, \omega \in V_{\mathbb{H}}^R$ and $\mathfrak{q} \in \mathbb{H}$, the inner product

$$\langle \cdot \mid \cdot \rangle : V_{\mathbb{H}}^R \times V_{\mathbb{H}}^R \longrightarrow \mathbb{H}$$

satisfies the following properties:

(i) $\overline{\langle \phi \mid \psi \rangle} = \langle \psi \mid \phi \rangle$
(ii) $\|\phi\|^2 = \langle \phi \mid \phi \rangle > 0$ unless $\phi = 0$, a real norm
(iii) $\langle \phi \mid \psi + \omega \rangle = \langle \phi \mid \psi \rangle + \langle \phi \mid \omega \rangle$
(iv) $\langle \phi \mid \psi\mathfrak{q} \rangle = \langle \phi \mid \psi \rangle\mathfrak{q}$
(v) $\langle \phi\mathfrak{q} \mid \psi \rangle = \overline{\mathfrak{q}}\langle \phi \mid \psi \rangle$