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The non-commutative topology of two-dimensional dirty superconductors

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ABSTRACT

Non-commutative analysis tools have successfully been applied to the integer quantum Hall effect, in particular for a proof of the stability of the Hall conductance in an Anderson localization regime and of the bulk-boundary correspondence. In this work, these techniques are implemented to study two-dimensional dirty superconductors described by Bogoliubov–de Gennes Hamiltonians. After a thorough presentation of the basic framework and the topological invariants, Kubo formulas for the thermal, thermoelectric and spin Hall conductance are analyzed together with the corresponding edge currents.

1. Introduction

Topological insulators have been the object of intense experimental and theoretical investigations over the last decade, and more recently also in the mathematical physics community. Many of the analytical tools developed for the study of integer quantum Hall systems could be adapted and extended to these novel topological systems. Several theoretical elements have become common themes of the field, in particular the identification of the topological invariants, their link to non-dissipative response coefficients and the bulk-boundary correspondence. For recent reviews and a vast literature, we refer to [1,2]. In our opinion, a thorough mathematical treatment of physical phenomena in topological superconductors is lacking to date and this work aims to partially fill this gap in dimension two.

Just as a large part of the physics literature, we focus on quadratic fermionic Hamiltonians described by an effective Bogoliubov–de Gennes (BdG) Hamiltonian on a particle–hole Hilbert space. These operators typically do not have particle and charge conservation due to the presence of a non-trivial pairing potential. Numerous such potentials are of physical interest and are reviewed in Section 2. Some of them have further symmetries, like a U(1) or SU(2) rotational invariance in the spin degrees of freedom. Such symmetries can be reduced out in a way that is independent of the dimension of physical space. While this symmetry analysis is somewhat standard by now, *e.g.* [3–5], it is nevertheless included in Section 2 for sake of completeness. In Section 3 it is then combined with the C^*-algebraic framework of [1,6,7] for the description of homogeneous media. Up to this point, the treatment is independent of the dimension of physical space, but starting from Section 4 we decided to restrict to two-dimensional tight-binding systems for which the main topological invariant is then the Chern number of the BdG Fermi projection. There is no principle difficulty in transposing also the numerous (strong and weak) invariants as discussed in [1,8] to BdG Hamiltonians in other dimensions (that is actually already covered by [9]), but carrying this out would have made this paper too encyclopedic and we hope that the most important concepts are communicated on the example of two-dimensional systems. On the other hand, we do go beyond earlier papers because a

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description of disordered systems is entirely covered. Merely an Anderson localization condition as in [1,7] has to be satisfied for the existence of the BdG Chern numbers.

The second part of the paper, that is Sections 5 to 9, then deals with various physical effects linked to a non-trivial topological invariant. In fact, it is shown how the bulk invariant (Chern number) determines the non-dissipative Hall response coefficient for the mass transport, the charge and spin transport (provided charge and spin are conserved), as well as the low temperature behavior of the thermal Hall transport. The latter so-called thermal quantum Hall effect [10,11] is probably the most interesting case, as no further symmetry has to be imposed on the BdG Hamiltonian (which is hence in the Cartan–Altland–Zirnbauer (CAZ) Class D). Unfortunately, we were unable to give a rigorous derivation of the Kubo formula for the thermal Hall conductance and hence merely analyze the (presumably correct) formula from [12–14]. The spin quantum Hall effect [15] is, on the other hand, merely a superposition of several (conventional) quantum Hall effects in each of the eigenspaces of the spin operator. The bulk-boundary correspondence (BBC) for the mass transport is a direct consequence of earlier results [1,16]. The BBC for the charge and spin transport is then a special case under supplementary symmetry constraints. Based on the physical interpretation of the boundary currents in these cases, we then derive a formula for the thermal boundary currents which, by the BBC, is then again quantized with a coefficient given by the bulk invariant. In our opinion, this considerably clarifies previous treatments [10,17,18]. As a final comment, let us mention that another physical effect in Class D are the Majorana zero modes attached to half-flux vortices [19]. This is not dealt with here, but in our previous paper [20].

2. Generalities on BdG Hamiltonians

2.1. BdG Hamiltonian in tight-binding representation

Let us begin by presenting the two-dimensional tight-binding BdG Hamiltonian which provides an effective description of electrons (quasi-particles) in a superconductor. Each single particle will be described by a one-particle Hamiltonian h acting on the one-particle Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^L$. Depending on what is to be described, the fiber $\mathbb{C}^L$ may contain a factor $\mathbb{C}^{2s+1}$ to describe the spin $s \in \mathbb{N}/2$ of the particle, and a fiber $\mathbb{C}^2$ to model a bipartite lattice structure like for the hexagon lattice, as well as any further local internal degrees of freedom of the particle, like simply several participating orbitals at every site. For sake of simplicity, we only consider the case $\mathbb{C}^L = \mathbb{C}^{2s+1}$ here. Of course, the case of spin $\frac{1}{2}$ is physically most relevant. In bra–ket notation, h is of the form

$$h \;=\; \sum_{n,n'\in\mathbb{Z}^2} \sum_{l,l'=1}^{L} h_{l,l'}(n,n')\,|n,l\rangle\langle n',l'| \;=\; \sum_{n,n'\in\mathbb{Z}^2} |n\rangle\, h(n,n')\,\langle n'|\;. \tag{1}$$

Here $h_{l,l'}(n,n')$ are complex numbers such that $h = h^*$ and $h(n,n') = (h_{l,l'}(n,n'))_{l,l'=1,\ldots,L}$ is an $L \times L$ matrix. Furthermore, $|n\rangle$ is the partial isometry to (spin) states at $n \in \mathbb{Z}^2$. The first main assumption on h will be that it is of finite range R, namely $h(n,n') = 0$ for $|n-n'| > R$. From a mathematical point of view, this locality condition could be somewhat relaxed, but this is irrelevant for the physics. Later on, the second main assumption is that h is space homogeneous (see Section 3 for details). This does allow h to contain a random potential, for example.

Next let us (canonically) second quantize h to an operator $\mathbf{h}$ on the fermionic Fock space $\mathcal{F} = \mathcal{F}(\mathcal{H})$ associated to $\mathcal{H} = \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^L$. If the creation and annihilation operators in the state $|n,l\rangle$ are denoted by $\mathfrak{c}^*_{n,l}$ and $\mathfrak{c}_{n,l}$, then the second quantization of (1) leads to

$$\mathbf{h} \;=\; \sum_{n,n'\in\mathbb{Z}^2} \sum_{l,l'=1}^{L} h_{l,l'}(n,n')\,\mathfrak{c}^*_{n,l}\mathfrak{c}_{n',l'} \;=\; \sum_{n,n'\in\mathbb{Z}^2} \mathfrak{c}^*_n\, h(n,n')\,\mathfrak{c}_{n'} \;=\; \mathfrak{c}^*\, h\, \mathfrak{c}\;.$$

Here, in the second formula $\mathfrak{c}_n$ denotes a vector (spinor) $\mathfrak{c}_n = (\mathfrak{c}_{n,l})_{l=1,\ldots,L}$ of annihilation operators, while in the third formula $\mathfrak{c}$ is the vector $\mathfrak{c} = (\mathfrak{c}_n)_{n\in\mathbb{Z}^2}$. There is no summability associated to this vector and it is merely used in order to have the compact notation $\mathbf{h} = \mathfrak{c}^* h \mathfrak{c}$ for the summation given above. The second quantized operator $\mathbf{h}$ clearly commutes with the number operator $\mathbf{N} = \mathfrak{c}^*\mathfrak{c}$ so that it conserves the particle number. Furthermore, let us point out that the anti-commutation relations for the $\mathfrak{c}$'s imply, with h^T denoting the transpose of h,

$$\mathfrak{c}^*\, h\, \mathfrak{c} \;=\; -\,\mathfrak{c}\, h^T\, \mathfrak{c}^* \;+\; \mathrm{Tr}(h)\,\mathbf{1}_{\mathcal{F}} \;=\; -\,\mathfrak{c}\,\overline{h}\,\mathfrak{c}^* \;+\; \mathrm{Tr}(h)\,\mathbf{1}_{\mathcal{F}}\;,$$

as long as $\mathrm{Tr}(h)$ is finite (*e.g.* for a finite sublattice of $\mathbb{Z}^2$). The term $\mathrm{Tr}(h)\,\mathbf{1}_{\mathcal{F}}$ is merely a constant shift in energy which will be neglected in the spirit of renormalization, even if $\mathrm{Tr}(h)$ is infinite as it gives no contribution to commutators. This is equivalent to working with a second quantized

$$\mathbf{h} \;=\; \frac{1}{2}\,\mathfrak{c}^*\, h\, \mathfrak{c} \;-\; \frac{1}{2}\,\mathfrak{c}\,\overline{h}\,\mathfrak{c}^*\;. \tag{2}$$