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The cohomological nature of the Fu–Kane–Mele invariant

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ABSTRACT

In this paper we generalize the definition of the FKMM-invariant introduced in De Nittis and Gomi (2015) for the case of "Quaternionic" vector bundles over involutive base spaces endowed with free involution or with a non-finite fixed-point set. In De Nittis and Gomi (2015) it has already be shown how the FKMM-invariant provides a cohomological description of the Fu–Kane–Mele index used to classify topological insulators in class AII. It follows that the FKMM-invariant described in this paper provides a cohomological generalization of the Fu–Kane–Mele index which is applicable to the classification of protected phases for other type of topological quantum systems (TQS) which are not necessarily related to models for topological insulators (e.g. the two-dimensional models of adiabatically perturbed systems discussed in Gat and Robbins, 2017). As a byproduct we provide the complete classification of "Quaternionic" vector bundles over a big class of low dimensional involutive spheres and tori.

1. Introduction

In its simplest incarnation, a *Topological Quantum System* (TQS) is a continuous matrix-valued map

$$X \ \ni \ x \ \longmapsto \ H(x) \ \in \ \mathrm{Mat}(\mathbb{C}^d) \tag{1.1}$$

defined on a topological space X sometimes called *Brillouin zone*. Although the precise definition of TQS requires some more ingredients and can be stated in a more general form [1,3,4], one can certainly state that the most relevant feature of these systems is the nature of the spectrum which is made by continuous (energy) bands. It is exactly the family of eigenprojectors emerging from this peculiar band structure which may encode information that are of topological nature. The study and the classification of the topological properties of TQS have recently risen to the level of "hot topic" in mathematical physics because its connection with the study of *topological insulators* in condensed matter (we refer to the two reviews [5] and [6] for a modern overview about topological insulators and an updated bibliography of the most relevant publications on the subject). However, systems like (1.1) are ubiquitous in the mathematical physics and are not necessarily linked with problem coming from condensed matter.

For the sake of simplicity we consider here a very specific and simple realization of a TQS:

$$H(x) \ := \ \sum_{j=0}^{2N} F_j(x)\ \Sigma_j. \tag{1.2}$$

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The $\{\Sigma_0, \ldots, \Sigma_{2N}\} \in \mathrm{Mat}(\mathbb{C}^{2^N})$ define a (non-degenerate) irreducible representation of the *complex Clifford algebra* $\mathrm{Cl}_{\mathbb{C}}(2N+1)$ and the real-valued functions $F_j : X \to \mathbb{R}$, with $j = 0, 1, \ldots, 2N$, are assumed to be continuous. Under the *gap condition*

$$Q(x) := \sum_{j=0}^{2N} F_j(x)^2 > 0 \tag{1.3}$$

one can associate the negative (or positive) part of the band spectrum of $x \mapsto H(x)$ with a complex vector bundle $\mathscr{E} \to X$ of rank 2^{N-1}, called *spectral bundle*.[1] For the details of this identification we refer to Section 5 as well as to [9, Section IV] or [3, Section 2] (the specific model (1.2) is considered in [10, Section 4]). The considerable consequence of the duality between gapped TQS's and spectral bundles is that one can classify the possible topological phases of the TQS by means of the elements of the set $\mathrm{Vec}^{2^{N-1}}_{\mathbb{C}}(X)$ given by the isomorphism classes of all rank 2^{N-1} vector bundles over X. Therefore, in this general setting, the classification problem for the topological phases of a TQS like (1.2) can be traced back to a classic problem in topology which has been elegantly solved in [11]. For instance, in the (physically relevant) case of a *low dimensional* base space X, the complete classification of the topological phases of (1.2) is provided by

$$\mathrm{Vec}^{2^{N-1}}_{\mathbb{C}}(X) \overset{c_1}{\simeq} H^2\big(X, \mathbb{Z}\big), \qquad \dim(X) \leqslant 3 \tag{1.4}$$

where on the right-hand side one has the second singular cohomology group of X and the map c_1 is the *first Chern class*.

The problem of the classification of the topological phases becomes more interesting, and challenging, when the TQS is constrained by the presence of certain symmetries or *pseudo-symmetries*. Among the latter, the *time-reversal symmetry* (TRS) attracted recently a considerable interest in mathematical physics community. A system like (1.2) is said to be time-reversal symmetric if there is an *involution* $\tau : X \to X$ on the base space and an *anti*-unitary map Θ such that

$$\begin{cases} \Theta\, H(x)\, \Theta^* = H\big(\tau(x)\big), & \forall\, x \in X \\ \Theta^2 = \epsilon\, \mathbb{1}_{2^N} & \epsilon = \pm 1. \end{cases} \tag{1.5}$$

The case $\epsilon = +1$ corresponds to an even (or bosonic) TRS. In this event, the spectral vector bundle $\mathscr{E}$ turns out to be equipped with an additional structure named *"Real"* by M. F. Atiyah in [12]. Therefore, in the presence of an even TRS the classification problem of the topological phases is reduced to the study of the set $\mathrm{Vec}^{2^{N-1}}_{\mathfrak{R}}(X, \tau)$ of isomorphism classes of rank 2^{N-1} vector bundles over X endowed with a "Real" structure. This problem has been analyzed and solved in [3]. In particular, in the low dimensional case one has that

$$\mathrm{Vec}^{2^{N-1}}_{\mathfrak{R}}(X, \tau) \overset{c_1^{\mathfrak{R}}}{\simeq} H^2_{\mathbb{Z}_2}\big(X, \mathbb{Z}(1)\big), \qquad \dim(X) \leqslant 3. \tag{1.6}$$

Here, in the right-hand side there is the equivariant Borel cohomology with local coefficient system $\mathbb{Z}(1)$ of the involutive space (X, τ) (see Section 2.3 and references therein for more details) and the isomorphism is given by *first "Real" Chern class* $c_1^{\mathfrak{R}}$ introduced for the first time by B. Kahn in [13] (see also [3, Section 5.6]). Note the close similarity between the Eqs. (1.4) and (1.6).

The case $\epsilon = -1$ describes an odd (or fermionic) TRS. Also in this situation the spectral vector bundle $\mathscr{E}$ acquires an additional structure called *"Quaternionic"* (or symplectic [14]) and the topological phases of a TQS with an odd TRS turn out to be labeled by the set $\mathrm{Vec}^{2^{N-1}}_{\mathfrak{Q}}(X, \tau)$ of isomorphism classes of rank 2^{N-1} vector bundles over X with "Quaternionic" structure.

The study of systems with an odd TRS is more interesting, and for several reasons also harder, than the case of an even TRS. Historically, the fame of these fermionic systems begins with the seminal papers [15,16] by L. Fu, C. L. Kane and E.J. Mele. The central result of these works is the interpretation of a physical phenomenon called *Quantum Spin Hall Effect* as the evidence of a non-trivial topology for TQS constrained by an odd TRS. Specifically, the papers [15,16] are concerned about the study of systems like (1.2) (with $N = 2$) where the base space is a torus of dimension 2 or 3 endowed with a "time-reversal" involution. These involutive *TR-tori* are the quotient spaces $(\mathbb{R}/\mathbb{Z})^d$ endowed with the quotient involution defined on $\mathbb{R}$ by $x \to -x$ (we denote these spaces with $\mathbb{T}^{0,d,0}$ according to a general notation which will be clarified in Eq. (1.10)). The distinctive aspect of the spaces $\mathbb{T}^{0,d,0}$ is the existence of a *fixed point set* formed by 2^d isolated points. The latter plays a crucial role in the classification scheme proposed in [15,16] where the different topological phases are distinguished by the signs that a particular function $\mathfrak{d}_{\mathscr{E}}$ defined by the spectral bundle (essentially the inverse of a normalized Pfaffian) takes on the 2^d points fixed by the involution. These numbers are usually known as *Fu–Kane–Mele indices*.

In the last years the problem of the topological classification of systems with an odd TRS has been discussed with several different approaches. As a matter of fact, many (if not almost all) of these approaches focus on the particular cases $\mathbb{T}^{0,2,0}$ and $\mathbb{T}^{0,3,0}$ with the aim of reproducing in different way the $\mathbb{Z}_2$-invariants described by the Fu–Kane–Mele indices. From one hand there are classification schemes based on *K-theory* and *KK-theory* [17–23] or *equivariant homotopy* techniques [24,25] which are extremely general. In the opposite side there are constructive procedures based on the interpretation of the

[1] Spectral bundles obtained as a result of a Bloch–Floquet transform, are sometimes called *Bloch bundle* [7,8].