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T-duality and the bulk-boundary correspondence

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ABSTRACT

String-theoretic T-duality can be exploited to simplify some features of the bulk-boundary correspondence in condensed matter theory. This paper surveys how T-duality links position and momentum space pictures of that correspondence.

0. Introduction

This paper reviews how T-duality, borrowed from string theory, simplifies some features of the bulk-boundary correspondence in condensed matter theory. This application of T-duals was first suggested and exploited by Mathai and Thiang [1–3], and was later extended in joint work with the author [4,5], and for details of the applications we refer to those papers.

After a short account of the historical context for this work in Section 1, Section 2 presents the bulk-boundary correspondence in position space, whilst Section 3 gives the momentum space perspective more suited to electron band theory. In Section 4 Cartier's lattice representation of the canonical commutation relations motivates the appearance of T-duality. The noncommutative geometric version of this duality is described in more detail in Section 5, and is then shown, in Section 6, to link the position and momentum versions of the bulk-boundary correspondence. Finally there is a brief discussion about whether H-flux, important in string theoretic T-duality, might also appear in solid state theory. Two Appendices cover relevant background aspects of C*- algebras and noncommutative geometry.

1. Groups, algebras, and topology in solid state physics

Symmetries played a crucial role in the rapid evolution of the fledgling solid state theory into its modern form following the 1925–6 discovery of a "new quantum theory" by Heisenberg and Schrödinger, [6–8]. Wigner (partly with von Neumann) showed that symmetries preserving transition probabilities must be described by unitary or antiunitary operators providing a projective unitary–antiunitary representation, or, in his terminology, a projective corepresentation $g \mapsto D(g)$ of a symmetry group G, with $D(g)D(h) = \sigma(g,h)D(gh)$, for some $\sigma(g,h) \in \mathbb{C}$ of modulus 1 [9,10]. Time reversal provides a key example of a symmetry represented antiunitarily. Further exploiting his insights in the 1950s, Wigner classified systems into three types [11–13]. Dyson later illuminated this result, by observing that the commuting algebra of an irreducible corepresentation D is, by Schur's Lemma, a real division algebra, and, by Frobenius' Theorem, this must be $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (the quaternions), [14]. (Wigner's original method is described in [10, §26].) Dyson showed further that the same argument applied both to the whole group G, and to the normal subgroup G_u of elements g such that $D(g)$ is unitary, and that,

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surprisingly, the irreducibles for G and G_u could independently have a commuting algebra $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, giving, in all, nine possibilities:

$$\begin{aligned} &\mathbb{RR},\ \mathbb{RC},\ \mathbb{RH},\\ &\mathbb{CR},\ \mathbb{CC},\ \mathbb{CH},\\ &\mathbb{HR},\ \mathbb{HC},\ \mathbb{HH}. \end{aligned}$$

Finally, he observed that the central $\mathbb{CC}$ possibility could come in two distinct forms, leading to ten classes in all.

It was unclear whether all ten of Dyson's possibilities could be realised experimentally. However, there were examples of the three Gaussian matrix ensembles, each also having a chiral version, giving six classes. Then, after three decades of limited activity, Zirnbauer [15] with Altland [16,17] took the idea up again and found four additional Gaussian ensembles based on the study of quantum dots. They also made explicit the precise correspondence between the classes of Gaussian ensembles, and Cartan's classification of symmetric spaces, [15,17], thus hinting at a geometric as well as a group-theoretic classification. (Three particular classes of symmetric spaces had appeared in [18, §V], but with the express regret that "a more illuminating" insight was lacking.)

After that there followed a decade of steadily accelerating activity, with important contributions by numerous authors, culminating Kitaev's synthesis, [19], which provided an explicit link to homotopy, K-groups, and symmetric spaces. (Surveys doing more justice to the numerous crucial papers are to be found in the review of [20], and, from a slightly different perspective, in [21]. The reviews by Freed and Moore [22], from a topological perspective, and by Prodan and Schulz-Baldes [23], from an operator algebraic slant, also include further developments since Kitaev's paper. Other recent strands are the use of (crystal lattice) equivariant K-theory, [24], and the topological investigation of Fermi arcs in Weyl semimetals, [25,26].)

The summary of the scheme which emerged can be summarised in the following table (named by Hasan and Kane the "Altland–Zirnbauer classification").

Cartan	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

The first column in this table gives the Cartan symmetric space corresponding to the physical situation, the next indicates whether time reversal Θ is absent, 0, or present with $\Theta^2 = \pm 1$, as indicated. The third column indicates particle–hole interchange symmetry with $\Xi^2 = \pm 1$, or its absence indicated by 0. The fourth column indicates the presence or absence of chirality with 1 or 0, respectively. It is clear that the first two rows in the table consist simply of repeated copies of the first pair of entries, with these interchanged between the first and the second row, whilst the other eight rows are obtained from the third row by cyclic permutations modulo 8. The entries $\mathbb{Z}_2$ are for presence or absence of e.g. superconducting phase, $\mathbb{Z}$ a quantised observable such as transverse Hall conductivity in the first row (whilst 0 indicates a default phase of the material). The periodicities 2 and 8 correspond precisely to those of complex and real K-theories, respectively.

To understand why K-theory could be relevant to condensed matter systems one has only to go back to Bloch's pioneering study of periodic systems such as crystals, [6]. Let $V \cong \mathbb{R}^d$ denote the group of spatial translations in $\mathbb{R}^d$, $\widehat{V}$ its Pontryagin dual, $L \cong \mathbb{Z}^d$ the subgroup of translations through the crystal lattice, $\widehat{L} \cong \mathbb{T}^d$ its dual, and $L^\perp = \{\xi \in \widehat{V} : \xi(\ell) = 1 \forall \ell \in L\}$ the reciprocal lattice. For the rest of this section we concentrate on $d = 3$. Spatial translations through the crystal lattice must commute with a Hamiltonian $H = P^2/(2m) + \Phi(Q)$ with periodic potential Φ. The action of the lattice group $L \cong \mathbb{Z}^d$ of translations on the Hilbert space $\mathcal{H} \cong L^2(V)$ of the quantum system provides a direct integral decomposition $\mathcal{H} = \int^{\oplus} \mathcal{H}(k)dk$ where $T(\ell)$ acts on $\mathcal{H}(k)$ as multiplication by $\chi_k(\ell)$ for $\chi_k \in \widehat{L}$. (Physicists tend to think of k as an element of $\widehat{V}$ which is periodic with respect to the reciprocal lattice $L^\perp$, as follows from the isomorphism $\widehat{L} \cong \widehat{V}/L^\perp$, and write $\chi_k(\ell) = \exp(ik.\ell)$, where the isomorphic vector groups $\widehat{V}$ and V are identified and given the usual Euclidean inner product denoted by a dot. Sometimes $\widehat{L}$ is considered a Brillouin zone, [27], though the Brillouin zones are usually introduced as subsets of $\widehat{V}$ which project onto $\widehat{L} = \widehat{V}/L^\perp$, and which, together, cover $\widehat{V}$, up to null sets. The first Brillouin zone consists of elements of $\widehat{V}$ which are closer to 0 than to any other lattice point $\lambda \in L^\perp$ in the Euclidean norm metric. Bellissard's noncommutative Brillouin zone is a crossed product C*-algebra, which allows for disorder, [28,29].)

General features can be illuminated within each space $\mathcal{H}(k)$, where, as Bloch realised, they become much simpler. In particular, the spectrum of the restriction $H(k)$ of the Hamiltonian to $\mathcal{H}(k)$ is discrete, and the ground state energy $E_0(k)$ is nondegenerate, [30, Vol 1, Ch VI.6,7], [31, Ch XIII.16, Th. XIII.89], and a lower bound for the gap $E_1 - E_0$ between the ground state and first excited state energy is known, [32, Th.2.1]. It is also known that $\mathcal{H}(k)$ depends continuously on k, giving a band structure to the energy levels. Since only a finite number of energy levels lie below the Fermi level we can assume that