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Spectrum of the Dirac operator on manifold with asymptotically flat end

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ABSTRACT

The spectrum of the Dirac operator is investigated on a Riemannian manifold one of whose end is asymptotically flat in some sense. It is proved that there is no non-zero eigenvalue and the essential spectrum is equal to $\mathbb{R}$.

1. Introduction

Let us consider a noncompact spin Riemannian manifold M and the Dirac operator $\mathcal{D}$ on M. If M is complete, then $\mathcal{D}$ has a unique self-adjoint extension [1] and the spectrum $\sigma(\mathcal{D})$ is well-defined. Since $\sigma(\mathcal{D})$ has several components in general, it is interesting to investigate the relation between geometric property of M and types of components of $\sigma(\mathcal{D})$(see for example [2], Chapter 7).

Similar problems for the Laplacians acting on functions on complete Riemannian manifolds were investigated by [3] and [4] in detail. Their main results show that if an end of a manifold is asymptotically hyperbolic or flat in some sense, the spectrum of the Laplacian consists of only essential spectrum. To prove this, [3] and [4] derive growth estimates at infinity of eigenfunctions by a method which was used for the Schrödinger operators on the Euclidean space. Hence geometric conditions are assumed only on an end. In this point they are different from preceding papers which require global assumptions ([5–8], etc.).

Concerning the Dirac operator, the result of [9] that the point spectrum $\sigma_p(\mathcal{D})$ of the real hyperbolic space is empty was later deduced only from the structure of the end [10]. Hence we study in this paper the spectrum $\sigma(\mathcal{D})$ when one of ends is asymptotically flat in some sense, and prove that $\sigma(\mathcal{D})$ consists of $\sigma_{\mathrm{ess}}(\mathcal{D}) = \mathbb{R}$ and possibly of eigenvalue 0, where $\sigma_{\mathrm{ess}}(\mathcal{D})$ denotes the essential spectrum.

Let us consider a noncompact complete spin Riemannian manifold (M, g) of dimension n. Following the terminology of [3,4], we say an end V of M has radial coordinates if and only if there exists an open subset U of M with the following property: V is a connected component of $M\backslash U$ whose boundary ∂V is compact, connected and the outward pointing normal exponential map $\exp^{\perp}_{\partial V} : N^{+}(\partial V) \to V$ induces a diffeomorphism.

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If an end V of M has radial coordinates, we denote by r the distance function on V from ∂V, and use the following notations for $0 \leq s < t$:

$$B(s,t) = \{x \in V \mid s < r(x) < t\},$$
$$B(s,\infty) = \{x \in V \mid s < r(x)\},$$
$$S(t) = \{x \in V \mid r(x) = t\}.$$

Also we denote the Riemannian measure of (M, g) by $\mathrm{d}v_g$, and the measure on each $S(t)$ induced from $\mathrm{d}v_g$ by $\mathrm{d}A$.

Theorem 1. *Let (M, g) be a noncompact complete spin Riemannian manifold of dimension $n \geq 3$ such that one of its ends has radial coordinates. Suppose that for a positive constant r_0, there exist constants A, B and C with $0 < A \leq 1 \leq B$, $B < \frac{n+1}{n-1}A$ and $C \geq 0$ such that*

$$\frac{A}{r}|X|^2 \leq \nabla \mathrm{d}r(X,X) \leq \frac{B}{r}|X|^2 \tag{1}$$

for $X \in (\nabla r)^\perp$ on $B(r_0,\infty)$,

$$|K| \leq \frac{C}{r^2} \quad \text{on} \quad B(r_0,\infty),$$
$$\frac{\partial \kappa}{\partial r} \leq 0 \quad \text{or} \quad \frac{\partial \kappa}{\partial r} = o\left(\frac{1}{r}\right) \quad \text{on} \quad B(r_0,\infty),$$

where K and κ denote the sectional curvature and the scalar curvature of M respectively. Then $\mathcal{D}$ has no non-zero eigenvalue.

Remark. On the Euclidean space with the standard metric, the distance function $\bar{r}$ from the origin satisfies $(\nabla \mathrm{d}\bar{r})(X,X) = \frac{1}{\bar{r}}|X|^2$ for $X \in (\nabla \bar{r})^\perp$. Let us define $V = \{x \in \mathbb{R}^n \mid \|x\| \geq R_*\}$. Then the distance function r on V from ∂V satisfies $r = \bar{r} - R_*$ and $(\nabla \mathrm{d}r)(X,X) = \frac{1}{r+R_*}|X|^2$. Hence if an end of M is isometric to V, then we may take $A \fallingdotseq 1$, $B \fallingdotseq 1$ and $C = 0$ with sufficiently large r_0.

Remark. As is stated in [4], the comparison theorem shows that properties of the radial curvature K_{rad} imply the inequalities (1).

Theorem 2. *Let (M, g) be a noncompact complete spin Riemannian manifold of dimension $n \geq 3$ such that one of its ends has radial coordinates. Suppose that for a positive constant r_1, there exist constants A_1, B_1 and C_1 with $0 < A_1 \leq B_1 \leq A_1 + \frac{1}{n-1}$, $(3n-5)B_1 < (3n-3)A_1 - 1$ and $C_1 \geq 0$ such that*

$$\frac{A_1}{r}|X|^2 \leq \nabla \mathrm{d}r(X,X) \leq \frac{B_1}{r}|X|^2 \tag{2}$$

for $X \in (\nabla r)^\perp$ on $B(r_1,\infty)$, and the sectional curvature K satisfies

$$|K| \leq \frac{C_1}{r^2}$$

on $B(r_1,\infty)$. Then the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{D}) = \mathbb{R}$.

Remark. If an end is Euclidean, we can take $A_1 \fallingdotseq 1$, $B_1 \fallingdotseq 1$, and $C_1 = 0$. Also the inequalities (2) follow from the properties of the radial curvature.

We shall show in the third section, $0 \in \sigma_p(\mathcal{D})$ for some metric on $\mathbb{R}^n$ which coincides with the standard metric outside a compact set. On the other hand, $\sigma_p(\mathcal{D}) = \emptyset$ for the standard metric on $\mathbb{R}^n$. Thus we have $\sigma = \sigma_p \cup \sigma_{\mathrm{ess}} = \emptyset \cup \mathbb{R}$ or $\sigma = \sigma_p \cup \sigma_{\mathrm{ess}} = \{0\} \cup \mathbb{R}$ for asymptotically flat spin manifolds.

One may consider the same problem for a spin manifold one of whose ends is asymptotically hyperbolic (the sectional curvature tends to -1). With appropriate assumptions, $\sigma_{\mathrm{ess}}(\mathcal{D}) = \mathbb{R}$ can be proved similarly. It is probable that the application of the method above implies $\sigma_p(\mathcal{D}) = \emptyset$.

2. Absence of non-zero eigenvalue

From Proposition 2.3 in [3], we get

$$-\frac{\partial}{\partial r}(\triangle r) = |\nabla \mathrm{d}r|^2 + \mathrm{Ric}(\nabla r, \nabla r).$$