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Weaving properties of generalized continuous frames generated by an iterated function system

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ABSTRACT

In this paper, we present some classes of generalized continuous weaving frames. It is shown that if the sets of lower frame bounds of discrete frames for a Hilbert space are bounded below, then the corresponding generalized continuous frames are woven. Necessary and sufficient conditions for generalized continuous weaving frames generated by an iterated function system are given.

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1. Introduction

Introduced by Bemrose, Casazza, Gröchenig, Lammers and Lynch, the notion of discrete *weaving frames* for separable Hilbert spaces appeared for the first time in [1]. The concept of weaving frames is motivated by distributed signal processing. Two discrete frames $\Phi = \{\phi_i\}_{i\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$ for a separable Hilbert space $\mathbb{H}$ are said to be woven, if there are universal positive constants A and B such that for every subset $\sigma \subset I$, the family $\{\phi_i\}_{i\in\sigma} \cup \{\psi_i\}_{i\in\sigma^c}$ is a frame for $\mathbb{H}$ with lower and upper frame bounds A and B, respectively. Weaving frames have potential applications in wireless sensor networks that require distributed processing under different frames, as well as pre-processing of signals using Gabor frames. To understand this: if Φ and Ψ are two sensor networks then we are interested in whenever parts of one network can be used to replace parts of the other. In the recent work by Bemrose et al. [1] the sensors are modeled by inner product with a vector in a frame or Riesz basis for a Hilbert space. Casazza, Freeman and Lynch [2] modeled these sensors by evaluation by linear functionals associated with an approximate Schauder frame or Schauder basis in separable Banach spaces. Casazza and Lynch reviewed the fundamental properties of weaving frames in [3]. Deepshikha and Vashisht [4] proved various necessary and sufficient conditions for infinitely woven discrete frames for separable Hilbert spaces and extended the concept of discrete weaving Hilbert space frames to continuous weaving frames in [5]. Some fundamental properties of vector-valued weaving frames can be found in [6]. It is also proved in [6] that if a family of vector-valued frames is woven, then the corresponding family of frames for atomic spaces is woven. However, the converse may not be true.

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Ali et al. in [7] and independently Kaiser [8] generalized the standard theory of discrete frames in Hilbert spaces to families of basic building blocks indexed by a measure space and called them *continuous frames*. Gabardo and Han in [9] called these frames "*frames associated with measurable spaces*". A continuous frame for a Hilbert space is a family of vectors indexed by some measure space which allows reconstruction of arbitrary elements by continuous superpositions. Continuous frames in mathematical physics are referred to as *coherent states*. For applications and development of continuous frames in various directions, see [8–13] and references therein. Ding introduced and studied generalized continuous frames constructed by using an iterated function system (IFS) in [14].

Notable contribution in the paper includes a class of generalized continuous woven frames for the space $L^2([-\pi, \pi])$ which has been obtained by using exponential frames and a family of continuous functions on a compact measure space, see Theorem 3.6. In Theorem 3.10, we use a family of square integrable functions in place of continuous functions to generate generalized continuous woven frames for the underlying space. Besides this we make use of iterated function systems (IFS), which is a system of contractions on $\mathbb{R}$, to generate generalized continuous woven frames (see Theorem 4.2).

Overview of the paper: Section 2 introduces some basic facts about discrete frames, weaving frames and generalized continuous frames in Hilbert spaces to make the paper self-contained. We introduce generalized continuous weaving frames in Hilbert spaces in Section 3. Sufficient conditions for generalized continuous weaving frames in terms of the positive lower bounds of the sets of lower frame bounds of discrete frames for a Hilbert space are given. Also, some classes of generalized continuous weaving frames are obtained. Finally, in Section 4, necessary and sufficient conditions for generalized continuous weaving frames by using an iterated function system are given.

2. Preliminaries

Throughout the paper $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ denote the set of all natural numbers, integers, real numbers and complex numbers, respectively. The set I is at most countable. The difference of two sets $U, V \subset \mathbb{R}$ is the set $U - V = \{a - b : a \in U, b \in V\}$. By χ_U we denote the characteristic function of the set U. We start by recalling the definition of a discrete frame in separable Hilbert spaces.

2.1. Discrete Hilbert frames

A countable sequence $\{f_k\}_{k\in I}$ in a separable complex Hilbert space $\mathbb{H}$ is called a *discrete Hilbert frame* (or *frame*) for $\mathbb{H}$ if there exist scalars $0 < a_o \leq b_o < \infty$ such that

$$a_o\|f\|^2 \leq \sum_{k\in I} |\langle f, f_k\rangle|^2 \leq b_o\|f\|^2 \text{ for all } f \in \mathbb{H}. \tag{2.1}$$

The scalars a_o and b_o are called *lower* and *upper frame bounds*, respectively. If $\{f_k\}_{k\in I}$ satisfies the upper inequality in (2.1), then we say that $\{f_k\}_{k\in I}$ is a *Bessel sequence* with *Bessel bound* b_o. If it is possible to choose $a_o = b_o$, then we say that the frame $\{f_k\}_{k\in I}$ is *tight*.

Concerning the evolution of the notion of Hilbert frames, it is necessary to mention the nobel books [15,16] and beautiful research tutorials [17,18] on basics and applications of frames in different directions in applied mathematics.

2.2. Weaving frames:

For a fixed $m \in \mathbb{N}$, we write

$$[m] = \{1, 2, \ldots., m\}.$$

Definition 2.1 (*[1]*)**.** A family of frames $\{\phi_{ij}\}_{i\in I, j\in[m]}$ for a separable Hilbert space $\mathbb{H}$ is said to be *woven*, if there are universal constants A and B such that for every partition $\{\sigma_j\}_{j\in[m]}$ of I the family $\{\phi_{ij}\}_{i\in\sigma_j, j\in[m]}$ is a frame for $\mathbb{H}$ with lower and upper frame bounds A and B, respectively.

Definition 2.2 (*[1]*)**.** A family of frames $\{\phi_{ij}\}_{i\in\mathbb{N}, j\in[m]}$ for $\mathbb{H}$ is (weakly) *woven* if for every partition $\{\sigma_j\}_{j\in[m]}$ of $\mathbb{N}$, the family $\{\phi_{ij}\}_{i\in\sigma_j, j\in[m]}$ is a frame for $\mathbb{H}$.

Remark 2.3. Bemrose et al. proved in [1] that this weaker form of weaving (given in Definition 2.2) is equivalent to weaving.

A characterization of weaving frames (that does not require universal frame bounds) and weaving Riesz bases can be found in [1]. Bemrose et al. proved sufficient conditions for weaving frames by means of perturbation theory and diagonal dominance. They also proved a geometric characterization of woven Riesz bases in terms of distance between subspaces of a Hilbert space. Casazza and Lynch [3] gave new properties of weaving frames. They discussed a weaving equivalent of an unconditional basis for weaving Riesz bases. Casazza, Freeman and Lynch [2] extended the concept of weaving Hilbert space frames to the Banach space setting. They introduced and study *weaving Schauder frames* in Banach spaces. It is proved in [2]

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