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Elastica as a dynamical system

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ABSTRACT

The elastica is a curve in $\mathbf{R}^3$ that is stationary under variations of the integral of the square of the curvature. Elastica is viewed as a dynamical system that arises from the second order calculus of variations, and its quantization is discussed.

1. Introduction

Ever since the beginning of the calculus of variations, second order problems such as the classical problem of the elastica have been considered. The peculiar situation that distinguishes most of the interesting examples in second order problems from the more familiar first order theory is that they are parametrization independent, and so the theory of such problems has a somewhat distinctive tone from that of the first order theory. A comprehensive review of this theory, as it was understood up until the 1960s, may be found in the monograph by Grässer [1].[1] By way of contrast, this paper seeks to understand how to systematically exploit the symmetry, conservation laws, and the associated first order canonical formalism, especially as they relate to the integration of the Euler–Lagrange equations of elastica.

A principal motivation for this paper was to understand that portion of the theory of second order variational problems that could reasonably be expected to be useful for elucidating the common behaviour of several geometric functionals on curves. Three examples that motivated our study are the elastica, the shape of a real Möbius band in terms of the geometry of the central geodesic [2], and the curve of least friction. Here we present elastica, and hope to report on the other two in due course.

For reasons not entirely clear to us, the geometric theory of higher order variational problems seems to have developed in a manner largely detached from the needs and concerns of concrete problems. This is a startling contrast to recent developments in geometric mechanics and their understanding of stability, bifurcation, numerical schemes, the incorporation of nonholonomic constraints, etc. The consequences of this are at least two-fold: first, it leads to a palpable sense of dread[2] when faced with trying to look up a formulation of some part of the theory that will cleanly explain how to compute something obvious, and second, a real disconnect between the theoretical insights and the actual computational methods. This disconnect is vividly illustrated in the problem of elastica.

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[1] This monograph is especially noteworthy for its comprehensive bibliography.

[2] This may be reduced to mere frustration by those less ignorant than the authors.

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Planar elasticae (the equilibrium shape of a linearly elastic thin wire) were considered at least as early as 1694 by James Bernoulli.[3] However, it was not until about 1742 that Daniel Bernoulli convinced Euler to solve the problem by using the isoperimetric method (the old name for the calculus of variations before Lagrange). From a variational point of view, the elastica is idealized as a curve that minimizes the integral over its length of the square of the curvature (that is, minimize $\int \kappa^2 \, ds$), and is thus naturally treated as a second order problem in the calculus of variations. Exhaustive results were then published by Euler in 1744 [5]. Since Euler's results were so comprehensive, it is not surprising that the study of elastica remained somewhat dormant until taken up again by Max Born in his thesis [6]. More recently a striking result was obtained in 1984 by Langer and Singer [7] when they demonstrated the existence of closed elastica that were torus knots. Their proof was noteworthy because they eschewed the usual variational machinery and employed clever *ad hoc* geometric arguments such as an adapted cylindrical coordinate system to aid their integration. In fact, a significant motivation for this paper was to see to what extent their techniques could be understood by a more pedestrian use of the second order calculus of variations that looked more like just 'turning the crank' on the variational machine, and thus had the comfort of familiarity of technique. Since it is not our intent to duplicate their calculations, but gain some group-theoretical insight into the integration procedure, we study a different problem where the arclength is not constrained.

Some features of the elastica problem instantly spring to mind in the modern geometrically oriented reader. The first is that the problem is manifestly invariant by the action of the Euclidean group. The second is that it would be very nice to have a theory that explained how to reduce the symmetry using the concomitant conservation laws that Emmy Noether taught us are in the problem, and then wind up with some form of reduced Euler–Lagrange equations. Assuming we can solve these reduced equations, and hence know the curvature and torsion of the elastic curve, we would expect a good theory to show us methods to determine the shape of our curve that go beyond a mere referral to the fundamental theorem of curves stating that the curvature and torsion of the curve determine it up to a Euclidean motion. Given all this, what we actually find when we look at the published work on elastica (such as [7] or [8]) is that it proceeds somewhat differently. In particular, almost none of the actual computations seem to follow any method that resembled the current theory. There are good reasons for this, and it is not due to ignorance of those geometers but a reflection that the theory at that time was presented in such a way as to simply be unhelpful, and unable to easily identify the geometric meaning of some of their calculations. This is the best explanation we have of the situation at the time and why it was still necessary a decade after [7] appeared for Foltinek (see [9]) to write a paper demonstrating the integration constants for elastica in terms of the conserved Noetherian momenta. Further work on symmetry and integration appeared in the article by Nesterenko and Scarpetta [10]. Another work that gave a detailed study of the conserved momenta was Coronado [11]. However, Coronado used the spatial coordinate x_1 as the parameter. Due to the parameter invariance, it is a valid procedure in the open dense domain in which $x_1 \neq 0$. This leads to a nonsingular Hamiltonian system to which the standard tools may be applied. In particular, the author analyzes for which values of the Noether invariants the reduced equations of motion can be integrated in the region where the solution exists. A disadvantage of Coronado's approach is that the choice of x_1 as the parameter obscures the geometric structure of the theory.

The plan of this paper is to first discuss the Euler–Lagrange equations for the elastica in arbitrary parametrization and the arclength parametrization. Relations between the conservation laws and the natural equations of the curve (the 'reduced equations', if you will) are derived. Then the conservation laws and symmetry group are systematically employed to integrate the equations. We then compare our approach to that of Langer and Singer in order to have an understanding of the appearance of the conserved quantities in the reduced equations as well as the role of a preferred subgroup of the Euclidean group in the integration. This yields a symmetry group theoretical explanation of the axis of the cylindrical coordinate system so cleverly (but mysteriously) employed by Langer and Singer. This is followed by a discussion of parametrization invariance and the Hamiltonian formalism. Elastica is then studied as a constrained Hamiltonian system which is invariant under the group SE(3) of rigid motions of space as well as the reparametrization group $\mathrm{Diff}_+ \mathbf{R}$ and obtain the corresponding momentum maps. We show that the constraint equations are equivalent to the vanishing of the $\mathrm{Diff}_+ \mathbf{R}$ momentum map $\mathscr{J}$, and the vanishing of the energy function. Reducing the $\mathrm{Diff}_+ \mathbf{R}$ symmetry of $\mathscr{J}^{-1}(0)$, we obtain a Hamiltonian system on the cotangent bundle of the unit sphere bundle over $\mathbf{R}^3$ with a single energy constraint. We solve the equations of motion for this system for every choice of initial data.

The paper concludes with the geometric quantization of elastica, and discusses the quantum representations of the groups SE(3) and $\mathrm{Diff}_+ \mathbf{R}$ as well as the quantum implementation of constraints.

It seems that it is a requirement that all authors on higher order calculus of variations have their own theoretical and notational preferences and foibles, and we are no exception to the rule. However, in order to spare the reader the tedium of wading through all of this before getting to the example of elastica, this material is summarized in the Appendix. Thus, the Appendix provides the necessary theoretical background of the second order variational calculus, especially as it pertains to symmetry and the Noether theory, as well as serving to fix notational conventions.

Finally, what is not in the paper. We do not explicitly construct SE(3) reduced spaces, nor do we discuss the Poisson bracket formalism. We also do not discuss the meaning of other groups, such as dilations, which, while not a symmetry group in the strict sense, are still of interest in understanding the structure of the solutions of the elastica.

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[3] See the delightful discussion by Levien in [3] or [4].