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On tadpole relations via Verdier specialization

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ABSTRACT

Using the construct of 'Verdier specialization', we provide a purely mathematical derivation of Chern class identities which upon integration yield the D3-brane tadpole relations coming from the equivalence between F-theory and associated weakly coupled type IIB orientifold limits. In particular, we find that all Chern class identities associated with weak coupling limits appearing in the physics literature are manifestations of a relative version of Verdier's specialization formula.

1. Introduction

Let $\varphi : Y \to B$ be an elliptic fibration over a smooth complete complex algebraic variety of arbitrary dimension whose total space Y is a smooth hypersurface in a $\mathbb{P}^2$-bundle $\pi : \mathbb{P}(\mathscr{E}) \to B$ given by a Weierstraß equation

$$Y : \left(y^2 = x^3 + fxz^2 + gz^3\right) \subset \mathbb{P}(\mathscr{E}).$$

The coefficients f and g are then sections of line bundles over B so that the fiber $\varphi^{-1}(b)$ over $b \in B$ is given by a Weierstraß equation with coefficients $f(b)$ and $g(b)$. The fibers of φ will degenerate to singular cubics over the discriminant hypersurface

$$\Delta : (4f^3 + 27g^2 = 0) \subset B,$$

with a generic fiber over Δ being a nodal cubic which will degenerate further to a cuspidal cubic over the codimension two locus

$$C : (f = g = 0) \subset B,$$

which in fact coincides with the singular locus of Δ in the case that the differentials df and dg are everywhere linearly independent.

In [1], string-theoretic arguments (i.e., S-duality between F-theory and associated weakly-coupled type IIB orientifold limits) led to the discovery of an interesting identity in the Chow group A_*B between the Chern–Schwartz–MacPherson (or simply CSM) class[1] of the constructible function $\mathbb{1}_\Delta + \mathbb{1}_C$ and the CSM class of a constructible function which seems to have no relation to $\mathbb{1}_\Delta + \mathbb{1}_C$, namely

$$c_{\mathrm{SM}}\left(\mathbb{1}_\Delta + \mathbb{1}_C\right) = c_{\mathrm{SM}}\left(2\mathbb{1}_O + \mathbb{1}_D - \mathbb{1}_S\right), \tag{1.1}$$

where the varieties O, D and S arise when viewing Y as a smooth deformation (parametrized by a disk in $\mathbb{C}$) of a certain singular variety Y_0. In the case that Y is a Calabi–Yau fourfold, the degree zero component of both sides of Eq. (1.1) yields a

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[1] We review the theory of CSM classes of constructible functions in Section 2.

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numerical relation predicted by string dualities which reflect the equivalence of D3-brane charge between F-theory and its weakly-coupled orientifold limit. Such relations equating charges between dual theories are often referred to in the physics literature as 'tadpole relations'.

In subsequent works [2–5], similar identities associated with other elliptic fibrations not in Weierstraß form were also arrived at via physical considerations. While the identities were indeed shown to hold in all these cases, a precise mathematical explanation as to why such identities should exist was lacking. In Remark 4.5 of [1], the construct known as 'Verdier specialization' – which associates constructible functions on a family over a disk with constructible functions on its central fiber – was used *a posteriori* to sketch a derivation of identity (1.1) solely from mathematical principles, but in an indirect manner that would have been unclear if the identity was not known to already hold in the first place. In any case, it was this example which motivated us to employ Verdier specialization to give a top-down approach to the explanation of the appearance of such identities. As such, we find all Chern class identities appearing in [1–5] to be manifestations of the following.

Theorem 1.1. *Let $\varphi : Y \to B$ be an elliptic fibration whose total space Y is a smooth zero-scheme of a section of a vector bundle $\mathscr{V} \to \mathbb{P}(\mathscr{E})$, and let s_t be a smoothly varying family of sections of $\mathscr{V} \to \mathbb{P}(\mathscr{E})$ over an open disk $\mathscr{D}$ about the origin in $\mathbb{C}$ whose zero-schemes give rise to a family $\mathscr{Y} \to \mathscr{D}$, such that the total space $\mathscr{Y}$ is smooth over $\mathscr{D} \setminus \{0\}$. Denote the central fiber of the family by $\varphi_0 : Y_0 \to B$. Then*

$$c_{\mathrm{SM}}\left(\varphi_* \mathbb{1}_Y\right) = c_{\mathrm{SM}}\left(\varphi_{0*}\sigma \mathbb{1}_{\mathscr{Y}}\right), \tag{1.2}$$

where $\varphi_* \mathbb{1}_Y$, $\varphi_{0*}\sigma \mathbb{1}_{\mathscr{Y}}$ are the pushforwards to B of the characteristic function $\mathbb{1}_Y$ and Verdier's specialization $\sigma \mathbb{1}_{\mathscr{Y}}$ of the characteristic function of the family $\mathscr{Y}$.

Theorem 1.1 is essentially a relative version of Verdier's formula (2.2), the proof of which we provide in Section 2. When Y is an elliptic fibration in Weierstraß form as defined earlier, computing both sides of Eq. (1.2) as given in Theorem 1.1 yields precisely identity (1.1) (as we show in Section 4). In what follows we review the theory of CSM classes of constructible functions along with the specialization morphisms first presented by Verdier, prove Theorem 1.1, discuss the physical motivation behind identities such as (1.1), and then show by explicit computation how Theorem 1.1 yields such Chern class identities.

2. CSM classes, Verdier specialization, and proof of Theorem 1.1

Let X be a complex variety. A constructible function on X is an integer-valued function of the form

$$\sum_i a_i \mathbb{1}_{W_i},$$

with each $a_i \in \mathbb{Z}$, $W_i \subset X$ a closed subvariety and $\mathbb{1}_{W_i}$ the function that evaluates to 1 for points inside of W_i and is zero elsewhere. The collection of all such functions forms an abelian group under addition, and is referred to as the *group of constructible functions* on X, denoted $F(X)$. A proper morphism $f : X \to Y$ induces a functorial group homomorphism $f_* : F(X) \to F(Y)$, which by linearity is determined by the prescription

$$f_* \mathbb{1}_W(p) = \chi\left(f^{-1}(p) \cap W\right), \tag{2.1}$$

where $W \subset X$ is a closed subvariety and χ denotes topological Euler characteristic with compact support. By taking $F(f) = f_*$, we may view F as a covariant functor from varieties to abelian groups. Another covariant functor from varieties to abelian groups is the homology functor H_*, which takes a variety to its integral homology. Motivated by a conjecture of Deligne and Grothendieck, in 1974 MacPherson explicitly constructed a natural transformation

$$c_* : F \to H_*,$$

such that for X smooth

$$c_*(\mathbb{1}_X) = c(TM) \cap [X] \in H_*X,$$

i.e., the total homological Chern class of X [6]. The class $c_*(\mathbb{1}_X)$ for arbitrary X is then a functorial generalization of Chern class to the realm of singular varieties. Moreover, such a class provides a means of generalizing the Gauß–Bonnet theorem to the singular setting, as functoriality implies

$$\int_X c_*(\mathbb{1}_X) = \chi(X),$$

where the integral sign denotes proper pushforward to a point. As the class $c_*(\mathbb{1}_X)$ was later shown by Brasselet and Schwartz to coincide with the Alexander-dual of a class constructed by Schwartz in the 1960s, we now refer to it as the *Chern–Schwartz–MacPherson (or simply CSM) class*. Moreover, for an arbitrary constructible function $\delta \in F(X)$ we will refer to $c_*(\delta)$ as the 'CSM class of δ', which will be denoted from here on by $c_{\mathrm{SM}}(\delta)$.