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Heat determinant on manifolds

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ABSTRACT

We introduce and study new invariants associated with Laplace type elliptic partial differential operators on manifolds. These invariants are constructed by using the off-diagonal heat kernel; they are not pure spectral invariants, that is, they depend not only on the eigenvalues but also on the corresponding eigenfunctions in a non-trivial way. We compute the first three low-order invariants explicitly.

1. Introduction

The heat kernel is one of the most important tools of global analysis, spectral geometry, differential geometry and mathematical physics, in particular, quantum field theory [1–7]. In quantum field theory the main objects of interest are described by the Green functions of self-adjoint elliptic partial differential operators on manifolds and their spectral invariants such as the functional determinants [8,5,9–11]. In spectral geometry one is interested in the relation of the spectrum of natural elliptic partial differential operators to the geometry of the manifold, more precisely, one studies the question: "To what extent does the spectrum of a differential operator determine the geometry of the underlying manifold?" [3,12–14].

There are also non-trivial links between the spectral invariants and the non-linear completely integrable evolution systems, such as Korteweg–de Vries hierarchy (see, e.g. [4]). In many interesting cases such systems are, in fact, infinite-dimensional Hamiltonian systems, and the spectral invariants of a linear elliptic partial differential operator are nothing but the integrals of motion of this system.

Instead of studying the spectrum of a differential operator directly one usually studies its spectral functions, that is, spectral traces of some functions of the operator, such as the zeta function, and the heat trace [11]. Usually one does not know the spectrum exactly; that is why, it becomes very important to study various asymptotic regimes. It is well known, for example, that one can get information about the asymptotic properties of the spectrum by studying the short time asymptotic expansion of the heat trace. The coefficients of this expansion, called the heat trace coefficients (or global heat kernel coefficients), play very important role in spectral geometry and mathematical physics [4,12,3,15,16].

The simplest case of a Laplace operator on a compact manifold without boundary is well understood and there is a vast literature on this subject, see [3,7,15,10] and the references therein. For a Laplace type operator on a compact manifold without boundary there is a well defined local asymptotic expansion of the heat kernel, which enables one to compute its diagonal and then the heat trace by directly integrating the heat kernel diagonal; this gives all heat trace coefficients. However, many ideas and techniques do not apply directly in more general cases.

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The existence of non-isometric isospectral manifolds demonstrates that the *spectrum alone does not determine the geometry* (see, e.g. [14,13]). That is why, we propose to study more general invariants of partial differential operators that are not spectral invariants, that is, they depend not only on the eigenvalues but also on the eigenfunctions, and, therefore, contain much more information about the geometry of the manifold.

In this paper we propose to study *new heat invariants* of second-order Laplace type elliptic partial differential operators acting on sections of vector bundles over Riemannian manifolds. Our goal is to develop a comprehensive methodology for such invariants in the same way as the theory of the standard heat trace invariants. Namely, we will define and study new heat invariants of differential operators and compute explicitly some leading terms of the asymptotic expansion of new heat invariants.

Our main result can be formulated as follows.

Theorem 1. *Let (M, g) be a smooth compact n-dimensional Riemannian manifold without boundary with metric g and $\mathcal{V}$ be a vector bundle over M of dimension N. Let ∇ be a connection on the vector bundle and Q be an endomorphism of the bundle $\mathcal{V}$. Let $\Delta = g^{\mu\nu}\nabla_\mu\nabla_\nu$ be the Laplace operator and $L : C^\infty(\mathcal{V}) \to C^\infty(\mathcal{V})$ be the Laplace type partial differential operator of the form $L = -\Delta + Q$. Let $U(t; x, x')$ be the heat kernel of the operator L,*

$$P_{\mu\nu'}(t; x, x') = \operatorname{tr} U^*(t; x, x')\nabla_\mu\nabla_{\nu'}U(t; x, x'), \tag{1.1}$$

where tr *is the fiber trace, and $K(t)$ be the functional defined by*

$$K(t) = \int_{M\times M} dx\, dx'\ \det P_{\mu\nu'}(t; x, x'). \tag{1.2}$$

Then there is an asymptotic expansion as $t \to 0$

$$K(t) \sim (4\pi)^{-n^2}\left(\frac{\pi}{2n}\right)^{n/2} t^{-n\left(n+\frac{1}{2}\right)}\sum_{k=0}^{\infty} t^k B_k, \tag{1.3}$$

where

$$B_k = \int_M dv\ b_k, \tag{1.4}$$

dv is the Riemannian volume element and b_k are differential polynomials in the Riemann curvature, the bundle curvature and the potential Q with some universal numerical coefficients that depend only on the dimensions n and N. The low order coefficients are

$$b_0 = \frac{1}{2}N^n, \tag{1.5}$$

$$b_1 = N^n\frac{12n^2 - n + 10}{72n}R - nN^{n-1}\operatorname{tr} Q, \tag{1.6}$$

$$\begin{aligned} b_2 \ =\ & N^n\left\{\frac{20n^4 - 8n^3 - 11n^2 - 6n + 6}{144n^2}R^2 + \frac{4n^3 + 11n^2 + n - 4}{120n^2}\nabla_\mu\nabla^\mu R\right.\\ &\left. + \frac{-24n^3 + 84n^2 - 576n + 385}{4320n^2}R_{\mu\nu}R^{\mu\nu} + \frac{8n^3 - 8n^2 - 18n + 15}{1440n^2}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}\right\}\\ & + \frac{n^3 - n^2 + 3n - 12}{12n^2}N^{n-1}\operatorname{tr}\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \frac{-12n^3 - 4n^2 + n + 2}{12n}N^{n-1}R\operatorname{tr} Q\\ & - \frac{n+2}{6}N^{n-1}\operatorname{tr}\nabla^\mu\nabla_\mu Q + nN^{n-1}\operatorname{tr} Q^2 + n(n-1)N^{n-2}(\operatorname{tr} Q)^2. \end{aligned} \tag{1.7}$$

Here $R_{\mu\nu\alpha\beta}$, $R_{\mu\nu}$ and R are the Riemann tensor, the Ricci tensor and the scalar curvature respectively, and $\mathcal{R}_{\mu\nu}$ is the curvature of the bundle connection.

Of course, the derivative terms can be neglected on manifolds without boundary.

This paper is organized as follows. In Section 2 we describe the necessary geometric framework and define the heat kernel and some invariants, such as the heat trace and the heat content. In Section 3. we define a new invariant called the heat determinant for scalar operators and show that on manifolds without boundary it is trivial. In Section 4 we define an alternative invariant that we call the heat determinant on vector bundles. In Section 5 we introduce the machinery of standard off-diagonal heat kernel asymptotics and compute the heat content asymptotics. In Section 6 we compute the determinant of the mixed derivatives of the heat kernel. In Section 7 we establish the asymptotics of the heat determinant and in Section 8 we compute some low-order coefficients of this expansion. Some of the technical formulas for the derivatives of the Synge function and the parallel transport operator are listed in the Appendix.