Journal of Geometry and Physics 104 (2016) 128–136

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

# Scalar curvature of spacelike hypersurfaces and certain class of cosmological models for accelerated expanding universes

Juan A. Aledo [a,*], Rafael M. Rubio [b]

[a] Departamento de Matemáticas, E.S.I. Informática, Universidad de Castilla-La Mancha, 02071 Albacete, Spain

[b] Departamento de Matemáticas, Campus de Rabanales, Universidad de Córdoba, 14071 Córdoba, Spain

ARTICLE INFO

*Article history:*
Received 8 July 2014
Accepted 8 February 2016
Available online 17 February 2016

*Keywords:*
Spacelike hypersurface
Scalar curvature
Generalized Robertson–Walker spacetime
Null convergence condition
Perfect fluid
Einstein's equations

ABSTRACT

We study the scalar curvature of spacelike hypersurfaces in the family of cosmological models known as generalized Robertson–Walker spacetimes, and give several rigidity results under appropriate mathematical and physical assumptions. On the other hand, we show that this family of spacetimes provides suitable models obeying the null convergence condition to explain accelerated expanding universes.



## 1. Introduction

In this paper we deal with the class of cosmological models called *generalized Robertson–Walker (GRW) spacetimes* (see Section 2), which are warped products $I \times_f F$ with base an open interval $(I, -dt^2)$ and fiber a Riemannian manifold $(F, g_F)$ whose sectional curvature is not assumed to be constant. Thus, our ambient spacetimes widely extend to those that are classically called Robertson–Walker (RW) spacetimes. Recall that the class of Robertson–Walker spacetimes includes the usual big-bang cosmological models, the de Sitter spacetime, the steady state spacetime, the Lorentz–Minkowski spacetime and the Einstein's static spacetime, among others. Unlikely to these spacetimes, our ambient spacetimes are not necessarily spatially-homogeneous. Note that being spatially-homogeneous, which is reasonable as a first approximation of the large scale structure of the universe, could not be appropriate when we consider a more accurate scale. Thus, a GRW spacetime could be a suitable spacetime to model a universe with inhomogeneous spacelike geometry [1]. On the other hand, small deformations of the metric on the fiber of classical Robertson–Walker spacetimes fit into the class of GRW spacetimes. Therefore, GRW spacetimes are useful to analyze if a property of a RW spacetime $\overline{M}$ is *stable*, i.e. if it remains true for spacetimes close to $\overline{M}$ in a certain topology defined on a suitable family of spacetimes [2]. In fact, a deformation $s \mapsto g_F^{(s)}$ of the metric of $F$ provides a one parameter family of GRW spacetimes close to $\overline{M}$ when $s$ approaches to 0. Note that a conformal change of the metric of a GRW spacetime with a conformal factor which only depends on $t$, produces a new GRW spacetime. Any GRW spacetime has a smooth global time function, and so it is *stably causal* [3, p. 64]. Moreover, if the fiber is complete then the GRW spacetime is *globally hyperbolic* [3, Th. 3.66]. On the other hand, if the fiber is compact then it is called *spatially closed*. In [4] the behavior of the geodesics of GRW spacetimes is studied.

* Corresponding author.
*E-mail addresses:* juanangel.aledo@uclm.es (J.A. Aledo), rmrubio@uco.es (R.M. Rubio).

http://dx.doi.org/10.1016/j.geomphys.2016.02.005

We will impose the spacetime to obey the *null convergence condition (NCC)*, which says that the Ricci tensor of the spacetime is semi-definite positive on every null (light-like) vector. Recall that the exact solutions to the Einstein equations with cosmological constant, provided that the stress–energy momentum tensor satisfies the weak energy condition, obey the null convergence condition.

On the other hand, the study of spacelike hypersurfaces in General Relativity is relevant for several questions, as foliations of spacetimes, change of expansion or contraction phases, the Cauchy problem for Einstein's equation, etc. (see, for instance, [5,6]). Moreover, for many problems in General Relativity, including the *Positive Mass Theorem* and the *Penrose Inequality*, knowledge of the entire spacetime is not necessary, rather attention may be focused solely on a spacelike hypersurface, playing the scalar curvature of this hypersurface an important role [7]. In addition, the choice of a constant mean curvature (CMC) spacelike hypersurface as initial data has been considered in order to deal with the Cauchy problem for Einstein's equation (see [8]).

In the first part of this paper (Section 3) we study the scalar curvature of spacelike hypersurfaces in a GRW spacetime which obeys the NCC (see Lemma 2). Thus, we obtain a general expression for the scalar curvature of an immersed spacelike hypersurface in such an ambient space (7), given several estimations when the spacetime obeys the NCC and characterizing those spacelike hypersurfaces which attain the equality in our estimations (Theorem 3 and Corollary 4). In this setting, we pay a special attention to the important case of maximal hypersurfaces (Corollaries 5 and 6). As a consequence of our results, in the particular case when the spacetime is the de Sitter space we provide a characterization of the totally umbilical spacelike hypersurfaces from a bound of the scalar curvature of the hypersurface (see Theorem 8 and Remark 9). We also particularize our study to the case of compact CMC hypersurfaces in a GRW spacetime which obeys the NCC, so obtaining more strong consequences including a Calabi–Bernstein type result (Theorem 11).

On the other hand, in the second part of the paper (Section 4) we apply our mathematical results to the study of a certain class of cosmological models, specifically GRW spacetimes filled with perfect fluid. In General Relativity one often employs a perfect fluid stress–energy momentum tensor to represent the source of the gravitational field. This fluid description is used where one assumes that the large-scale proprieties of the universe can be studied by assuming a perfect fluid description of the sources. A review of the specific literature shows that, in fact, almost all the cosmological studies use the perfect fluid model. We focus on the case where GRW spacetimes satisfying the NCC constitute perfect fluid models adequate to describe universes at dominant dark energy stage, namely, accelerated expanding universes. We end up particularizing our study to the family of spatially closed GRW spacetimes. In this setting, we are able to express the total energy on a compact spacelike hypersurface in terms of its scalar and mean curvatures (Theorems 13 and 15). Finally, in the simplest case of a 3-dimensional GRW spacetime, as a consequence of the Gauss–Bonnet theorem we provide a nice expression of the total energy in terms of the Euler characteristic of the surface, its mean curvature and its volume (Theorem 16).

## 2. Preliminaries

Let $(F, g_F)$ be an $n(\geq 2)$-dimensional (connected) Riemannian manifold, $I$ an open interval in $\mathbb{R}$ endowed with the metric $-dt^2$, and $f$ a positive smooth function defined on $I$. Then, the product manifold $I \times F$ endowed with the Lorentzian metric

$$\bar{g} = -\pi_I^*(dt^2) + f(\pi_I)^2 \, \pi_F^*(g_F), \tag{1}$$

where $\pi_I$ and $\pi_F$ denote the projections onto $I$ and $F$, respectively, is called a *Generalized Robertson–Walker (GRW) spacetime* with *fiber* $(F, g_F)$, *base* $(I, -dt^2)$ and *warping function* $f$. Along this paper we will represent this $(n+1)$-dimensional Lorentzian manifold by $\overline{M} = I \times_f F$.

The coordinate vector field $\partial_t := \partial/\partial t$ globally defined on $\overline{M}$ is (unitary) timelike, and so $\overline{M}$ is time-orientable. We will also consider on $\overline{M}$ the conformal closed timelike vector field $K := f(\pi_I)\,\partial_t$. From the relationship between the Levi-Civita connections of $\overline{M}$ and those of the base and the fiber [9, Cor. 7.35], it follows that

$$\overline{\nabla}_X K = f'(\pi_I)\, X$$

for any $X \in \mathfrak{X}(\overline{M})$, where $\overline{\nabla}$ is the Levi-Civita connection of the Lorentzian metric (1).

We will denote by $\overline{\mathrm{Ric}}$ and $\overline{S}$ the Ricci tensor and the scalar curvature of $\overline{M}$, respectively. It is a straightforward computation (see [9, Cor. 7.43]) to check that

$$\overline{\mathrm{Ric}}(X, Y) = \mathrm{Ric}^F(X^F, Y^F) + \left(\frac{f''}{f} + (n-1)\frac{f'^2}{f^2}\right)\overline{g}(X^F, Y^F) - n\frac{f''}{f}\overline{g}(X, \partial_t)\overline{g}(Y, \partial_t) \tag{2}$$

for $X, Y \in \mathfrak{X}(\overline{M})$, where $\mathrm{Ric}^F$ stands for the Ricci tensor of $F$. Here $X^F$ denotes the lift of the projection of the vector field $X$ onto $F$, that is,

$$X = X^F - \overline{g}(X, \partial_t)\partial_t. \tag{3}$$

Recall that a Lorentzian manifold $\overline{M}$ obeys the *Null Convergence Condition (NCC)* if its Ricci tensor $\overline{\mathrm{Ric}}$ satisfies $\overline{\mathrm{Ric}}(X, X) \geq 0$, for all null vectors $X \in \mathfrak{X}(\overline{M})$. If $\overline{M} = I \times_f F$, given a null vector field $X \in \mathfrak{X}(\overline{M})$ we get, by decomposing $X$ as