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On holomorphic Riemannian geometry and submanifolds of Wick-related spaces

Victor Pessers, Joeri Van der Veken *

KU Leuven, Department of Mathematics, Celestijnenlaan 200B - Box 2400, 3001 Leuven, Belgium

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ABSTRACT

In this article we show how holomorphic Riemannian geometry can be used to relate certain submanifolds in one pseudo-Riemannian space to submanifolds with corresponding geometric properties in other spaces. In order to do so, we shall first rephrase and extend some background theory on holomorphic Riemannian manifolds, which is essential for the later application of the presented method.

1. Introduction

In this article, we show how certain problems in (pseudo-)Riemannian submanifold theory, that are situated in different ambient spaces, can be related to each other by translating the problem to an encompassing holomorphic Riemannian space. Our approach seems new in the area of submanifold theory, although it incorporates several existing insights, such as the theory on analytic continuation, complex Riemannian geometry and real slices, as well as the method of Wick rotations, which is mainly used in physics.

The relation between pseudo-Riemannian geometry and complex analysis can be traced back to the very birth of pseudo-Riemannian geometry. In the early publications on Lorentzian geometry by Poincaré and Minkowski (cf. [1,2]), the fourth coordinate of space–time was represented as *it* (or *ict*), so that space–time was essentially modeled as $\mathbb{R}^3 \times i\mathbb{R}$, where the standard complex bilinear form played the role of metric. Likely due to a later reformulation by Minkowski himself, this point of view soon fell in abeyance in favor of the nowadays more common presentation in terms of an indefinite real inner product. Admittedly, as long as ones attention is kept restricted to four-dimensional Minkowski space alone, the use of complex numbers to deal with the signature bears little advantage.

This relationship between space–time geometry and complex numbers received renewed attention, when it was shown by Wick how problems from the Lorentzian setting are turned into problems in a Euclidean setting, after a so-called Wick

* Corresponding author.
E-mail addresses: victor.pessers@wis.kuleuven.be (V. Pessers), joeri.vanderveken@wis.kuleuven.be (J. Van der Veken).

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rotation is applied on the time coordinate (cf. [3]). This method of Wick rotations lays at the basis of the later theory on Euclidean quantum gravity, developed by Hawking and Gibbons among others (cf. [4]), but also in other areas of theoretical physics it remained a valuable tool ever since.

Despite that the concept of Wick rotations is known by physicists for quite some time already, there are still many research domains where such insights have not been fully exploited yet. With this article, we hope to demonstrate that also for the particular area of submanifold theory, one may benefit from taking a complex viewpoint to problems about submanifolds of pseudo-Riemannian spaces. It should be noted that the subject of complex Riemannian manifolds, first introduced in an also physically motivated article by LeBrun (cf. [5]), has appeared in several articles on submanifold theory (e.g. [6,7]).

The structure of this article is as follows. In Sections 2–4, we rephrase and extend the theory of holomorphic Riemannian manifolds from respectively a *complex-linear* perspective, a complex analytic perspective and subsequently the viewpoint of submanifold theory. This theory is then used in 5th and final section of this article, where we demonstrate by three different examples our method to relate certain kinds of submanifolds in one pseudo-Riemannian space, to submanifolds with corresponding geometric properties in other so-called Wick-related spaces.

2. Preliminaries on complex Riemannian geometry

2.1. Complex vector spaces with holomorphic inner product

In the following, we will assume all vector spaces to be finite dimensional. We adopt the following two definitions on subspaces of a complex vector space (cf. [8]).

Definition 2.1. Let V be a complex vector space. We call a real linear subspace $W \subset V$ *totally real* if $W \cap iW = \{0\}$ and *generic* if $W + iW = V$.

Observe that if W is both generic and totally real, then its real dimension equals the complex dimension of V.

Definition 2.2. A *holomorphic inner product space* is a complex vector space V equipped with a non-degenerate complex bilinear form g.

Given a holomorphic inner product on V one can always choose an orthonormal basis. This means that any n-dimensional holomorphic inner product space can be identified with $\mathbb{C}^n$, equipped with the standard holomorphic inner product

$$g_0(X, Y) = X_1Y_1 + \cdots + X_nY_n, \tag{2.1}$$

where $X = (X_1, \ldots, X_n) \in \mathbb{C}^n$ and $Y = (Y_1, \ldots, Y_n) \in \mathbb{C}^n$. The name 'holomorphic inner product' comes from the fact that, unlike the more used sesquilinear inner product, the above product is a holomorphic function from $\mathbb{C}^n \times \mathbb{C}^n$ to $\mathbb{C}$.

We note that, given a complex-linear operator $A : V \to V$, there does not necessarily exist a basis of V consisting of eigenvectors of A, not even when A is symmetric with respect to a holomorphic inner product on V. However, if there exists a basis of eigenvectors of a symmetric operator A, there also exists an orthonormal basis of eigenvectors of A.

Definition 2.3. Given a holomorphic inner product space (V, g), we use the term *real slice* to denote a real linear subspace $W \subset V$, for which $g|_W$ is non-degenerate and real valued, i.e., $g(X, Y) \in \mathbb{R}$ for all $X, Y \in W$.

Example 2.4. Consider the standard holomorphic inner product space $(\mathbb{C}^n, g_0)$ and let $\{e_1, \ldots, e_n\}$ be the standard basis. The real linear subspace

$$\mathbb{R}^n_k := \mathrm{span}_{\mathbb{R}}\{ie_1, \ldots, ie_k, e_{k+1}, \ldots, e_n\} \tag{2.2}$$

is a real slice. Indeed, the restriction of the inner product $g_0|_{\mathbb{R}^n_k}$ is real valued, in particular, it is the standard pseudo-Euclidean metric of signature k on $\mathbb{R}^n$, as the notation $\mathbb{R}^n_k$ suggests. If $z_j = x_j + iy_j$ are the standard coordinates on $\mathbb{C}^n$, the space $\mathbb{R}^n_k$ is given by $x_1 = \cdots = x_k = y_{k+1} = \cdots = y_n = 0$. Hence, $(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n)$ are natural real coordinates on $\mathbb{R}^n_k$.

Since any n-dimensional holomorphic inner product space (V, g) can be identified with $(\mathbb{C}^n, g_0)$ by choosing an orthonormal basis, we can always find a real slice of any signature k, with $0 \leq k \leq n$ in V, namely the one corresponding to $\mathbb{R}^n_k \subset \mathbb{C}^n$. One easily verifies that all real slice of V come from such an identification, in particular, they are all related by the action of the complex orthogonal group $\mathrm{O}(n, \mathbb{C})$ on V.

Proposition 2.5. *The real slices of a holomorphic inner product space are totally real subspaces.*

Proof. Let W be a real slice of a holomorphic inner product space (V, g), and let $X \in W \cap iW$. We need to show that $X = 0$. Since $X \in iW$, there exists a vector $X' \in W$ such that $X = iX'$. Then we have that for all $Y \in W$, both $g(X, Y)$ and $g(X', Y) = -ig(X, Y)$ are real. Hence $g(X, Y) = 0$ for all $Y \in W$, so that the non-degeneracy of g implies $X = 0$. □