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The local counting function of operators of Dirac and Laplace type

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ABSTRACT

Let P be a non-negative self-adjoint Laplace type operator acting on sections of a hermitian vector bundle over a closed Riemannian manifold. In this paper we review the close relations between various P-related coefficients such as the mollified spectral counting coefficients, the heat trace coefficients, the resolvent trace coefficients, the residues of the spectral zeta function as well as certain Wodzicki residues. We then use the Wodzicki residue to obtain results about the local counting function of operators of Dirac and Laplace type. In particular, we express the second term of the mollified spectral counting function of Dirac type operators in terms of geometric quantities and characterize those Dirac type operators for which this coefficient vanishes.

1. Introduction

Let M be a closed Riemannian manifold of dimension d and with metric g. Let E be a smooth complex hermitian vector bundle over M. As usual we denote by $C^\infty(M; E)$ the space of smooth sections of E, by $L^2(M; E)$ the Hilbert space of square integrable sections equipped with the natural inner product defined by the hermitian structure on the fibers and the metric measure μ_g on M.

A second order partial differential operator $P : C^\infty(M; E) \to C^\infty(M; E)$ is said to be of Laplace type if its principal symbol σ_P is of the form $\sigma_P(\xi) = g_x(\xi, \xi)\mathrm{id}_{E_x}$ for all covectors $\xi \in T^*_x M$. In local coordinates this means that P is of the form

$$P = -g^{ij}(x)\partial_i\partial_j + a^k(x)\partial_k + b(x), \tag{1.1}$$

where a^k, b are smooth matrix-valued functions, and we have used Einstein's sum convention. Given a Laplace type operator P, it is well known that there exist a unique connection ∇ on E and a unique bundle endomorphism $V \in C^\infty(M; \mathrm{End}(E))$ such that $P = \nabla^*\nabla + V$. Then P is called a generalized Laplace operator if $V = 0$. We will assume here that P is self-adjoint and non-negative. Thus there exists an orthonormal basis $\{\phi_j\}_{j=1}^\infty$ for $L^2(M; E)$ consisting of smooth eigensections such that $P\phi_j = \lambda_j^2\phi_j$, where λ_j are chosen non-negative and correspond to the eigenvalues of the operator $P^{1/2}$.

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A first order partial differential operator $D : C^\infty(M;E) \to C^\infty(M;E)$ is said to be of Dirac type if its square is of Laplace type. This means that the principal symbol σ_D of D satisfies the Clifford algebra relations

$$\sigma_D(\xi)\sigma_D(\eta) + \sigma_D(\eta)\sigma_D(\xi) = 2g_x(\xi,\eta)\mathrm{id}_{E_x} \quad (\xi,\eta \in T_x^*M). \tag{1.2}$$

Given a Dirac type operator D, we denote by γ the action of the Clifford algebra bundle on E generated by the principal symbol of D, and suppose ∇ is a connection on E that is compatible with the Clifford action γ (see Section 3 for details). We call the triple (E,γ,∇) a Dirac bundle, and can express D uniquely as $D = \gamma\nabla + \psi$, where $\psi \in C^\infty(M;\mathrm{End}(E))$ is called the potential of D associated with the Dirac bundle (E,γ,∇). In particular, D is called the generalized Dirac operator associated with the Dirac bundle (E,γ,∇) if $\psi = 0$. We also assume D is self-adjoint, which means there exists a discrete spectral resolution $\{\phi_j,\mu_j\}_{j=1}^\infty$ of D, where $\{\phi_j\}_{j=1}^\infty$ is an orthonormal basis for $L^2(M;E)$, and $D\phi_j = \mu_j\phi_j$ for each j. Obviously, ϕ_j will be eigensections of $P = D^2$ with eigenvalues μ_j^2. Therefore, using the notation from before $\lambda_j = |\mu_j|$.

Given a classical (polyhomogeneous) pseudodifferential operator A of order $m \in \mathbb{R}$ acting on sections of E, the microlocalized spectral counting function $N_A(\mu)$ of D is defined as

$$N_A(\mu) = \begin{cases} \displaystyle\sum_{0\le\mu_j<\mu} \langle A\phi_j,\phi_j\rangle & \text{if } \mu > 0, \\ \displaystyle\sum_{\mu\le\mu_j<0} \langle A\phi_j,\phi_j\rangle & \text{if } \mu \le 0. \end{cases} \tag{1.3}$$

Thus, $N_A(\mu)$ is a piecewise constant function on $\mathbb{R}$ such that

$$N_A'(\mu) = \sum_{j=1}^\infty \langle A\phi_j,\phi_j\rangle \delta_{\mu_j},$$

where δ_{μ_j} denotes the delta function on $\mathbb{R}$ centered at μ_j. In case A is the operator of multiplication by a function $f(x) \in C^\infty(M)$ then

$$N_A(\mu) = \int_M f(x) N_x(\mu)\, d\mu_g(x),$$

where

$$N_x(\mu) = \begin{cases} \displaystyle\sum_{0\le\mu_j<\mu} \|\phi_j(x)\|_{E_x}^2 & \text{if } \mu > 0, \\ \displaystyle\sum_{\mu\le\mu_j<0} \|\phi_j(x)\|_{E_x}^2 & \text{if } \mu \le 0 \end{cases} \tag{1.4}$$

is the so-called local counting function of D.

Let $\chi \in \mathscr{S}(\mathbb{R})$ be a Schwartz function such that the Fourier transform $\mathscr{F}\chi$ of χ is 1 near the origin and $\mathrm{supp}(\mathscr{F}\chi) \subset (-\delta,\delta)$, where δ is a positive constant smaller than half the radius of injectivity of M. It is well known (see e.g. [1–6] for various special cases) that

$$(\chi * N_A')(\mu) = \sum_{j=1}^\infty \langle A\phi_j,\phi_j\rangle \chi(\mu-\mu_j) \sim \sum_{k=0}^\infty \mathscr{A}_k(A,D)\mu^{d+m-k-1} \quad (\mu\to\infty). \tag{1.5}$$

This can be derived from studying the Fourier integral operator representation of $A\frac{\mathrm{Sign}(D)+\mathrm{Id}_E}{2}e^{-it|D|}$ via the stationary phase method. The mollified spectral counting coefficients $\mathscr{A}_k(A,D)$ do not depend on the choice of χ, and are locally computable in terms of the total symbols of $A\frac{\mathrm{Sign}(D)+\mathrm{Id}_E}{2}$ and $|D|$. Note also the corresponding expansion for $\mu \to -\infty$ can be easily obtained from replacing D with $-D$:

$$(\chi * N_A')(\mu) \sim \sum_{k=0}^\infty \mathscr{A}_k(A,-D)|\mu|^{d+m-k-1} \quad (\mu\to-\infty). \tag{1.6}$$

Therefore, the function $\chi * N_A'$ contains all the information about $\{\mathscr{A}_k(A,\pm D)\}_{k=0}^\infty$.

One of the purposes of the paper is to show how to explicitly determine $\mathscr{A}_k(A,D)$. In particular, for any bundle endomorphism F of E, we can express $\mathscr{A}_1(F,D)$ in terms of geometric quantities such as g,γ,∇,ψ,F. To compare, Sandoval [5] obtained an explicit expression of $\mathscr{A}_1(\mathrm{Id}_E,D)$, while Branson and Gilkey [7] can also do so for $\mathscr{A}_1(F,D)$ whenever F is of the form $f\mathrm{Id}_E$ where f is a smooth function on M. In the case of more general first order systems with $F = \mathrm{Id}_E$ a formula was also more recently obtained by Chervova, Downes and Vassiliev [8]. In the case of operators of Dirac-type our paper also explains the relation between the results of [8] and the subsequent [9] on one hand and known heat-trace invariants on the other.

The mollified spectral counting coefficients $\mathscr{A}_k(F,D)$ closely relate to the local counting function of D. Recall D is a self-adjoint Dirac type operator with spectral resolution $\{\phi_j,\mu_j\}_{j=1}^\infty$. For each j we denote by $\Psi_j = \Psi_j(x,y)$ the Schwartz