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Implicit Lagrange–Routh equations and Dirac reduction

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ABSTRACT

In this paper, we make a generalization of Routh's reduction method for Lagrangian systems with symmetry to the case where not any regularity condition is imposed on the Lagrangian. First, we show how implicit Lagrange–Routh equations can be obtained from the Hamilton–Pontryagin principle, by making use of an anholonomic frame, and how these equations can be reduced. To do this, we keep the momentum constraint implicit throughout and we make use of a Routhian function defined on a certain submanifold of the Pontryagin bundle. Then, we show how the reduced implicit Lagrange–Routh equations can be described in the context of dynamical systems associated to Dirac structures, in which we fully utilize a symmetry reduction procedure for implicit Hamiltonian systems with symmetry.

1. Introduction

There is no doubt that there exists a close relation between symmetries and conservation laws, which has been one of the fundamental motivations for many geometric approaches to mechanical systems. The symmetry group of a dynamical system can always be used to reduce the system to one with fewer variables. When symmetry, besides, leads to conserved quantities, it can be very advantageous to incorporate that property into the reduction process. For example, when the system is Hamiltonian on a symplectic manifold, one first restricts the attention to the submanifold determined by the conserved momenta, and only later one takes the quotient of this submanifold by the remaining symmetry (which in general happens to be only a subgroup of the original symmetry group). This, in a few words, is the so-called *symplectic reduction theorem* (see, [1,2]).

Symplectic reduction may be applied to the standard case of classical Hamiltonian systems defined on the cotangent bundle. While this procedure has been thoroughly studied in the literature, its Lagrangian counterpart, the so-called Routh or tangent bundle reduction, has traditionally received much less attention, even though since its conception in [3,4] it has proven to be a valuable tool to obtain and discuss, e.g., the stability of steady motions or relative equilibria. A few modern approaches to the topic can be found in the papers [5–7]. One of the drawbacks of these papers is that a regularity condition needs to be assumed. In this paper, we will focus upon Routh reduction within the context of Dirac structures, *without* assuming any regularity hypotheses.

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For simplicity, let us consider for a moment the case of a Lagrangian $L(x, \dot{x}, \dot{\theta})$ with a single cyclic coordinate θ. The first step in Routh's procedure is to write the corresponding velocity $\dot{\theta}$ in terms of the remaining coordinates and velocities $(x, \dot{x})$ by making use of the conservation law $\partial L/\partial\dot{\theta} = \mu$, which follows from Noether's theorem. One then introduces the restriction $R^\mu(x, \dot{x})$ of the function $L - \dot{\theta}(\partial L/\partial\dot{\theta})$ to the level set where the momentum is μ, the so-called *Routhian* (see, e.g., [2]). With this function, one can observe that the remaining Euler–Lagrange equations of the coordinates x, again when constrained to the level set associated to μ, are in fact Euler–Lagrange equations for the Routhian R^μ. The end result of Routh's reduction method is therefore that it reduces the Euler–Lagrange equations of the Lagrangian $L(x, \dot{x}, \dot{\theta})$ to those of $R^\mu(x, \dot{x})$ on a reduced configuration space. A crucial ingredient in the above process, however, is that the Lagrangian satisfies the regularity condition $(\partial^2 L/\partial^2\dot{\theta}) \neq 0$, which is necessary for carrying out the first step. Routh's procedure and the regularity condition for it to be applicable can be generalized to arbitrary Lagrangians with a (possibly) non-Abelian symmetry group G. In this situation, the condition is often referred to as *G-regularity*.

It is easy to construct a Lagrangian which fails to be G-regular. The following example in $\mathbb{R}^2$ is taken from [8]:

$$L(x, y, v_x, v_y) = (v_x)^2 + v_x v_y - V(x).$$

Note that it has a cyclic coordinate y (and therefore an Abelian symmetry group $G = \mathbb{R}$), but that it is not G-regular. Also linear G-invariant Lagrangians will always fail to be G-regular. For example, the dynamics of N vortices in the plane admit the following Lagrangian:

$$L(z_l, \dot{z}_l) = \frac{1}{2i}\sum_k \gamma_k\left(\bar{z}_k\dot{z}_k - z_k\dot{\bar{z}}_k\right) - \frac{1}{2}\sum_n\sum_{k\neq n}\gamma_n\gamma_k \ln|z_n - z_k|, \quad z_l \in \mathbb{C},$$

where $\gamma_k \in \mathbb{R}$ are parameters of the model; see [9] for more details. This Lagrangian is clearly linear in its velocities and invariant under rotations of the vortices in the plane but not G-regular. Also in the context of plasma physics, linear Lagrangians often appear (see, e.g., [10]).

The aim of this paper is to extend Routh's method to the most general case where not any regularity condition is imposed on the Lagrangian. Our approach is based on the *Hamilton–Pontryagin principle* (as it is called in [11]) which leads to an implicit formulation of the Euler–Lagrange equations on the so-called Pontryagin bundle $TQ \oplus T^*Q$. We will show that under the assumption of symmetry, we can reduce these implicit equations to a set of *reduced implicit Lagrange–Routh equations*. The key ingredient is that we can circumvent the hypotheses on regularity by keeping the momentum constraint implicit throughout. Our method involves a generalized Routhian function which is defined on a certain submanifold of the Pontryagin bundle rather than on a submanifold of the tangent bundle, as is commonly the case for G-regular systems. Implicit Lagrangian systems can be geometrically described in the framework of Dirac structures (see [12]). The definition of Dirac structure in [13,14] was originally inspired by the notion of Dirac brackets, which was coined by Paul Dirac (in the 1950s) for dealing with constraints in the Hamiltonian setting when the given Lagrangian is singular (see e.g. [15,16]). So from the very start, there has been a strong relation with singular Lagrangians and constraints. In the second part of the paper, we will show that the reduced implicit Lagrange–Routh equations may be also formulated in terms of a Dirac structure, by considering a reduction method known for implicit Hamiltonian systems from [17,18].

For completeness, we mention that the paper by [8] also deals with the general case. However, these authors use a variational approach which is based on Hamilton's principle rather than on the Hamilton–Pontryagin principle. Therefore, it focuses on different aspects of the theory.

This paper is organized as follows. In Section 2, we review the derivation of the standard implicit Euler–Lagrange equations for a possibly degenerate (or singular) Lagrangian L via the Hamilton–Pontryagin principle. We use a technique that is similar to the one that has been used in, for instance, [5,19] to rewrite the implicit Euler–Lagrange equations in terms of an *anholonomic* frame. Once the implicit equations on a general frame are obtained, we specialize these expressions to a particular frame adapted to a given symmetry of the Lagrangian (Section 3). For a prescribed value of momenta, we find the implicit Lagrange–Routh equations, and express them in an invariant form. In Section 4 we reduce them to obtain the reduced implicit Lagrange–Routh equations. The regular cases are discussed in Section 5, where we illustrate our theory by showing how the reduced implicit Lagrange–Routh equations agree with those developed in the literature. Section 6 rephrases the previous results in terms of reduction of Dirac structures. We show how the reduced implicit Lagrange–Routh equations correspond to a certain reduced implicit Hamiltonian system. Finally, in Section 7, some examples are shown.

2. A version of the Hamilton–Pontryagin principle using anholonomic frames

Hamilton–Pontryagin principles. Let Q be a configuration manifold of a mechanical system with $\dim Q = n$. Coordinates on Q are given by q^α, fiber coordinates on TQ and T^*Q will be denoted by v^α and p_α, respectively. In the following, the index α runs from 1 to n unless otherwise noted. The notations are chosen in such a way that we can make a notational difference between a general curve $(q(t), v(t))$ in TQ, and the lifted curve $(q(t), \dot{q}(t))$ in TQ of a curve $q(t)$ in Q, where t denotes the time in $I = \{t \in \mathbb{R} \mid a \leq t \leq b\}$.

The Hamilton–Pontryagin principle leads to an implicit form of the Euler–Lagrange equations of a possibly degenerate Lagrangian L. These equations follow from considering the following variational principle on $TQ \oplus T^*Q$:

$$\delta\int_b^a \left[L(q, v) + \langle p, (\dot{q} - v)\rangle\right]dt = \delta\int_b^a \left[L(q^\alpha, v^\alpha) + p_\alpha(\dot{q}^\alpha - v^\alpha)\right]dt = 0,$$