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Taub–NUT dynamics with a magnetic field

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ABSTRACT

We study classical and quantum dynamics on the Euclidean Taub–NUT geometry coupled to an abelian gauge field with self-dual curvature and show that, even though Taub–NUT has neither bounded orbits nor quantum bound states, the magnetic binding via the gauge field produces both. The conserved Runge–Lenz vector of Taub–NUT dynamics survives, in a modified form, in the gauged model and allows for an essentially algebraic computation of classical trajectories and energies of quantum bound states. We also compute scattering cross sections and find a surprising electric–magnetic duality. Finally, we exhibit the dynamical symmetry behind the conserved Runge–Lenz and angular momentum vectors in terms of a twistorial formulation of phase space.

1. Introduction

The Euclidean Taub–NUT (TN) geometry has been studied extensively and from several different points of view. It is interesting as a particularly simple example of a gravitational instanton, it can be viewed as a Kaluza–Klein geometrisation of the Dirac monopole and it arises in the context of monopole moduli spaces, either directly, or, in its 'negative mass' form, as an asymptotic limit. In each of these contexts it plays a role akin to that of the hydrogen atom in atomic physics, both in the general sense of being the simplest example and in the technical sense of sharing simplifying features with the hydrogen atom, like a Runge–Lenz type conserved quantity.

The four-dimensional Maxwell equations on TN space have a simple source-free solution which is intimately connected to the TN geometry. This was first pointed out by Pope [1,2] who went on to show that the index of the Dirac operator minimally coupled to this Maxwell field is non-trivial. The index and the properties of zero-modes of this gauged Dirac operator were recently studied in detail in our paper [3] from which the current paper evolved.

The Maxwell field first considered by Pope has played an important role in various contexts. Its field strength is a harmonic two-form which is self-dual for an appropriate choice of orientation and also square-integrable. It is exact, with a globally defined gauge potential which is, however, not square-integrable. In other words, the harmonic two-form generates the non-trivial L^2-cohomology in the middle dimension of TN space. This is the reason why it was important in tests of S-duality on monopole moduli spaces [4,5].

One can relate the self-dual two-form directly to the TN geometry by noting that, with a suitable normalisation, it is the Poincaré dual of the $\mathbb{CP}^1$ which compactifies TN to $\mathbb{CP}^2$ [6]. More generally, one can understand the L^2-cohomology of TN and its multi-centre generalisation in terms of the ordinary cohomology of a suitable compactification [7].

For all of these reasons, it is not surprising that the inclusion of the self-dual gauge field in the dynamics on TN space turns out to be mathematically natural. In this paper, we consider the geodesic motion on TN coupled to the self-dual gauge field as our model for the classical dynamics and the Dirac and Laplace operators on TN, also minimally coupled to the gauge

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field, as quantum models. We show that all the interesting algebraic features of ordinary TN dynamics carry over to the gauged case, and that, moreover, bounded motions and quantum bound states, neither of which are possible on (regular, 'positive mass') TN alone, occur in the gauged dynamics.

In general qualitative and semi-quantitative terms, the classical and quantum dynamics on TN space is strikingly similar to the Kepler problem and the non-relativistic hydrogen atom. The basic reason why the four-dimensional TN geometry can model the three-dimensional motion of a charged particle is the fact that TN is a Kaluza–Klein geometrisation of the Dirac monopole in three dimensions [8,9]. The more detailed similarities are related to the conserved Runge–Lenz vector in both cases. In this sense, one can view our inclusion of the magnetic field in the TN dynamics as analogous to the inclusion of a magnetic monopole field in the much studied extension of the Kepler problem to the MICZ Kepler problem [10,11].

From the Kaluza–Klein point of view, the magnetic field of the monopole is already encoded in the geometry, and the name 'magnetic field' for an additional abelian gauge field on TN is potentially confusing. We adopt it here because it is justified from the four-dimensional point of view. As we shall see, the magnetic field leads to magnetic binding akin to that responsible for Landau levels in planar systems. In fact, in a limit where TN becomes flat Euclidean four-space, the four-dimensional magnetic field is constant, and the bound states that we find become ordinary Landau levels. This picture of magnetic binding also provides a qualitative explanation of the index found by Pope and of the form of the zero-modes discussed in [3].

We have organised our presentation to proceed from the most direct to more abstract treatments of gauged TN dynamics. We begin in Section 2 with a brief general discussion of how a magnetic field on a two-dimensional Riemannian manifold can produce bound states. In Section 3, we collect conventions for describing the TN geometry and the associated Dirac and Laplace operators coupled to a magnetic field. We turn to the classical dynamics in Section 4, discuss the conserved Runge–Lenz and angular momentum vectors of the gauged geodesic motion on TN and describe the classical trajectories. Section 5 contains a direct solution of the eigenvalue problem for the gauged Laplace operator on TN space through separation of variables. We exhibit the promised bound states, give their energies and degeneracies and compute scattering cross sections. In Section 6, we solve the quantum problem algebraically, using a quantum version of the Runge–Lenz vector. In Section 7, we exhibit the symmetry underlying the conservation of angular momentum and the Runge–Lenz vectors from a twistorial description of phase space. Our final Section 8 contains a brief discussion, our conclusions and an outlook onto open problems.

2. A toy model: motion on a surface with magnetic field

We can gain a qualitative understanding of bound states on TN coupled to a Maxwell field by considering a two-dimensional model, consisting of a two-dimensional manifold with metric and magnetic field. We will encounter a manifold and metric of the same kind in our study of TN as a geodesic submanifold, and the magnetic field as the restriction of the Maxwell field to the geodesic submanifold. However, here we study the two-dimensional model in its own right.

Consider a two-dimensional manifold diffeomorphic to an open disk D with $U(1)$-invariant metric of the form

$$ds^2 = dR^2 + c^2(R)d\gamma^2. \tag{2.1}$$

For consistency with our later discussion of the TN geometry we take the angular coordinate γ in the interval $[0, 4\pi)$, so that $4\pi c$ is the length of a $U(1)$ orbit. The radial coordinate R is the proper radial distance from the origin and has range $[0, \infty)$, and we assume a form of c near $R = 0$ to ensure that the metric is smooth there. We are interested in two kinds of behaviour of the function c.

The first case captures what happens in the regular TN geometry. The function c has the finite range $[0, L)$ for some positive real number L so that the length of the $U(1)$ orbits remain bounded. Moreover we assume that $c(0) = 0$ and that c is strictly monotonic, so that one can picture the metric as being induced on a cigar-shaped surface of revolution in three-dimensional Euclidean space, as shown in Fig. 1. The qualitative behaviour of geodesics on such a surface is well known and follows from Clairaut's relation. Generic geodesics spiral on the cigar. Geodesics spiralling towards the tip will be reflected at some point and spiral out. All geodesics ultimately move arbitrarily far away from the tip and there are no geodesics which remain in a region bounded by a finite value of R.

The second case captures what happens in the singular or 'negative mass' TN. The function c diverges at $R = 0$, has the range (L, ∞) and is monotonically decreasing. As an embedded surface, this is a funnel, with the opening at $R = 0$ and the tip at $R = \infty$ as shown in Fig. 1. Generic geodesics again spiral on this surface, but now there are two kinds of behaviour. Geodesics which travel straight down the funnel or spiral only slowly may escape to $R = \infty$. However, geodesics travelling into the funnel with sufficiently high angular momentum relative to their speed will bounce back and remain inside a region bounded by some finite value of R.

We now return to the first case with monotonically increasing $c \in [0, L)$ and consider the inclusion of a magnetic field of a specific type given by the two-form

$$B = d\left(\frac{pc^2}{2L^2}d\gamma\right) = \frac{p}{L^2}c\, dc \wedge d\gamma, \tag{2.2}$$