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Existence of an attractor and determining modes for structurally damped nonlinear wave equations

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HIGHLIGHTS

- 3D nonlinear wave equation with structural damping is studied.
- It is proved that there exists an exponential attractor of the semigroup generated by the problem.
- It is shown that asymtptic bahavior of solutions is determined by asymptotic behavior of finitely many their Fourier modes.

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ABSTRACT

The paper is devoted to the study of asymptotic behavior as $t \to +\infty$ of solutions of initial boundary value problem for structurally damped semi-linear wave equation $\partial_t^2 u(x,t) - \Delta u(x,t) + \gamma(-\Delta)^\theta \partial_t u(x,t) + f(u) = g(x)$, $\theta \in (0,1)$, $x \in \Omega$, $t > 0$ under homogeneous Dirichlet's boundary condition in a bounded domain $\Omega \subset \mathbb{R}^3$. We proved that the asymptotic behavior as $t \to \infty$ of solutions of this problem is completely determined by dynamics of the first N Fourier modes, when N is large enough. We also proved that the semigroup generated by this problem when $\theta \in (\frac{1}{2}, 1)$ possesses an exponential attractor.

1. Introduction

In the present paper we consider the following initial boundary value problem for structurally damped nonlinear wave equation:

$$\partial_t^2 u(x,t) - \Delta u(x,t) + \gamma(-\Delta)^\theta \partial_t u(x,t) + f(u) = g(x),$$
$$x \in \Omega,\ t > 0, \tag{1.1}$$

$$u(x,t) = 0,\ x \in \partial\Omega,\ t \geq 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \partial_t u(x,0) = u_1(x),\ x \in \Omega, \tag{1.3}$$

where $\gamma > 0, \theta \in [0,1)$ are given numbers, $\Omega \subset \mathbb{R}^3$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, u_0, u_1 are given initial functions, and $g \in L^2(\Omega)$ is a given source term and $f(\cdot)$ is a given nonlinear term.

The nonlinearity $f(\cdot) \in C^1(\mathbb{R})$ is assumed to satisfy the conditions

$$-C_1 + a_1|s|^q \leq f'(s) \leq C_2 + a_2|s|^q, \quad \forall s \in \mathbb{R} \tag{1.4}$$

for some constants $C_i, a_i > 0, i = 1, 2$, and $q > 0$.

The most famous representative of Eq. (1.1) is the so-called strongly damped nonlinear wave equation (i.e. Eq. (1.1) with $\theta = 1$):

$$\partial_t^2 u(x,t) - \Delta u(x,t) - \gamma \Delta \partial_t u(x,t) + f(u) = g(x). \tag{1.5}$$

This equation is used in modeling of a number of physical processes. It was used in the study of the motion of viscoelastic materials, e.g., in modeling the deviation from the equilibrium configuration of linearly viscoelastic solid with short memory, in the presence of an external force depending on the displacement. Equation of the form (1.5) is used in describing evolution of the current u in a Josephson junction. This kind of equation appears also in the Frémond theory for phase transitions when microscopic accelerations are taken into account. We refer to [1–3] and references therein for more details about the physical origins of Eq. (1.5).

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In [4] Foias and Prodi proved that if the difference between the first N modes of any two solutions of the 2D Navier–Stokes equations tends to zero as $t \to +\infty$, for N large enough then the difference between corresponding two solutions also tends to zero as $t \to +\infty$. Later on Ladyzhenskaya [5] proved that the semigroup generated by the initial boundary value problem for the 2D Navier–Stokes equations has a global attractor and if the projections of two trajectories on the attractor on the subspace of the phase space spanned on first eigenfunctions of the Stokes operator coincide then the corresponding trajectories also coincide. These pioneering works inspired further intensive study of finite-dimensional behavior for Navier–Stokes equations, reaction diffusion equation, Kuramoto–Sivashinsky equations, Cahn–Hilliard equation, damped nonlinear Klein–Gordon equation, damped Kirchhoff equations and other dissipative nonlinear PDE's (see, e.g., [4,6–22] and references therein). It was shown that these problems possess finite-dimensional global attractors, finite number of determining modes, determining nodes, determining volume elements and other finite number of determining parameters.

A number of works in this regard were devoted to initial boundary value problems for Eq. (1.5). Under the natural dissipativity assumption

$$\liminf_{|r|\to\infty} f'(r) \geq -\lambda_1, \tag{1.6}$$

and the growth restriction

$$|f'(r)| \leq C(1+|r|^4), \ \forall r \in \mathbb{R} \tag{1.7}$$

it was established that the semigroup generated by this problem has a finite dimensional global attractor in the phase space $H_0^1(\Omega) \times L^2(\Omega)$ (see, e.g., [3,11,16,23–33] and references therein). Existence of a global attractor in the phase space $H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega)$ was established just under the dissipativity condition (1.6) (see, e.g., [3]). Asymptotic behavior of solutions of the problem (1.1)–(1.3) is studied in [34], where the authors proved existence of a finite-dimensional global attractor of the semigroup associated to this problem for $\theta \in [\frac{1}{2}, 1]$ when the nonlinear term satisfies the dissipativity condition and growth condition (1.7).

It is well-known that by using the Galerkin method one can show that the problem (1.1)–(1.3) has a global weak energy solution $u \in L^\infty(0, T; H_0^1(\Omega) \cap L^{q+2}(\Omega))$ with $\partial_t u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^\theta(\Omega))$, and $u_{tt} \in L^\infty(0, T; H^{-1}(\Omega) \oplus L^{\frac{q+2}{q+1}}(\Omega))$ for each $\theta \in [0, 1]$, and any $q \geq 0$. Moreover, this problem has an absorbing ball in the phase space

$$\mathcal{E} := H_0^1(\Omega) \cap L^{q+2}(\Omega) \times L^2(\Omega).$$

Finally, we would like to note that the uniqueness of the weak energy solution of the problem (1.1)–(1.3) for this problem under the condition (1.4)

$$\text{for } q \in [0, \frac{8\theta}{3-4\theta}) \text{ when } \theta \in [\frac{1}{2}, \frac{3}{4}),$$

and

$$\text{for any } q \geq 0 \text{ when } \theta \in [\frac{3}{4}, 1],$$

as well as the existence of an exponential attractor for the case $\theta = 1$ with $q \geq 0$ is established in [35]. Moreover, in [26], the well-posedness and existence of a finite dimensional attractor for the same problem is established when $\theta \in (0, \frac{1}{2})$ and $q < 4$, and for the case when $\theta = \frac{1}{2}$ and $q = 4$ the existence of an exponential attractor is established in [29]. Recently, in [28], the existence of an exponential attractor of this semigroup is established in the case

$$\theta \in (\frac{1}{2}, \frac{3}{4}), \ \text{ and } q \in (0, \frac{8\theta}{3-4\theta}].$$

Our main goal in this paper is to show that the asymptotic behavior as $t \to +\infty$ of solutions of the problem (1.1)–(1.3) in $\mathcal{E}$

$$\begin{aligned} &\text{for } q \in [0, \frac{8\theta}{3-4\theta}] \text{ when } \theta \in (\frac{1}{2}, \frac{3}{4}), \text{ and} \\ &\text{for } q \geq 0 \text{ when } \theta \in [\frac{3}{4}, 1) \end{aligned} \tag{1.8}$$

is determined by asymptotic behavior as $t \to +\infty$ of projection of solution on some finite-dimensional subspace of the corresponding phase space. Furthermore, we also establish for the first time the existence of an exponential attractor in $\mathcal{E}$ for the case

$$q \geq 0 \text{ when } \theta \in [\frac{3}{4}, 1).$$

The results we obtained can be considered as a development of results on finite-dimensional asymptotic behavior of solutions of initial boundary value problem for 3D nonlinear wave equation with structural damping obtained in [26,29,35].

2. Preliminaries

Throughout the paper we will employ the following standard notations:

- $L^p(\Omega)$, $1 \leq p \leq \infty$, and $H^s(\Omega)$ are the usual Lebesgue and Sobolev spaces, respectively.
- $\mathcal{A} := -\Delta$ is the Laplace operator subject to the no-slip homogeneous Dirichlet boundary condition with the domain $H^2(\Omega) \cap H_0^1(\Omega)$. The operator $\mathcal{A}$ is a self-adjoint positively definite operator in H, whose inverse $\mathcal{A}^{-1}$ is a compact operator from $L^2(\Omega)$ into $L^2(\Omega)$. Thus it has an orthonormal system of eigenfunctions $\{w_j\}_{j=1}^\infty$ of $\mathcal{A}$.
- We denote by $\{\lambda_j\}_{j=1}^\infty$, $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, the eigenvalues of the operator $\mathcal{A}$ corresponding to orthonormal set of eigenfunctions $\{w_j\}_{j=1}^\infty$, repeated according to their multiplicities.
- For $N \geq 1$, P_N will denote the projection operator in $L^2(\Omega)$ onto the subspace generated by the first N eigenfunctions, and we set $Q_N = I - P_N$. $\|\cdot\|$, $(\cdot,\cdot)$ stand for the standard norm, and the inner product in $L^2(\Omega)$.
- $\|\cdot\|_s$ denotes the norm in $H^s := D(\mathcal{A}^{\frac{s}{2}})$, and $\|\cdot\|_{-s}$ denotes the norm in the dual space $H^{-s} := \Big(D(\mathcal{A}^{\frac{s}{2}})\Big)'$. Then, for $s \in \mathbb{R}$ one has the following Parseval identity:

  $$\|u\|_s^2 = \|\mathcal{A}^{\frac{s}{2}} u\|^2 = \sum_{k=1}^\infty \lambda_k^s (u, w_k)^2.$$

  From this it is easy to deduce the following Poincaré inequality. For all $s \geq r \in \mathbb{R}$, and $N \geq 1$ we have

  $$\|Q_N u\|_s^2 \geq \lambda_{N+1}^{s-r} \|Q_N u\|_r^2 \tag{2.1}$$

  for any $u \in H^s$.
- In the sequel we will make use of the following monotonicity inequality (see, e.g. [22])

  $$\begin{aligned} d_1\big(|u|^p + |v|^p\big)|u-v|^2 &\leq \big(|u|^p u - |v|^p v\big)\big(u-v\big) \\ &\leq d_2\big(|u|^p + |v|^p\big)|u-v|^2, \ p \geq 1, \end{aligned} \tag{2.2}$$

  for any $u, v \in \mathbb{R}$, where $d_1, d_2 > 0$ are constants depending only on p and the *Young inequality with* ε

  $$ab \leq \epsilon \frac{a^p}{p} + \frac{1}{\epsilon^{q/p}} \frac{b^q}{q}, \tag{2.3}$$

  which is satisfied for each $a, b > 0$ and $\epsilon > 0$, where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.