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FLUIDS, GEOMETRY, AND THE ONSET OF NAVIER-STOKES TURBULENCE IN THREE SPACE DIMENSIONS

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Dedicated to our friend Edriss Titi on the occasion of his 60th birthday

Abstract. A theory for the evolution of a metric g driven by the equations of three-dimensional continuum mechanics is developed. This metric in turn allows for the local existence of an evolving three-dimensional Riemannian manifold immersed in the six-dimensional Euclidean space. The Nash-Kuiper theorem is then applied to this Riemannian manifold to produce a *wild* evolving C^1 manifold. The theory is applied to the incompressible Euler and Navier-Stokes equations. One practical outcome of the theory is a computation of critical profile initial data for what may be interpreted as the onset of turbulence for the classical incompressible Navier-Stokes equations.

1. Introduction

The purpose of this paper is to continue our previous study of the link between the equations of inviscid continuum mechanics and the motion of Riemannian manifolds (*cf.* [1]). More specifically, we have shown in [1] that a solution of the system of balance laws of mass and momentum in two space dimensions can be mapped into an evolving two-dimensional Riemannian manifold in $\mathbb{R}^3$. Furthermore, it is shown that the geometric image of smooth solutions of the continuum equations for non-wild data (not simple shears) can be shadowed by a non-smooth geometric motion. In addition, the geometric initial value problem for these non-smooth solutions has an infinite number of energy preserving non-smooth solutions. Since the earlier paper [1] was focused on two-dimensional continuum mechanics, it is natural to develop a theory to deal with three space dimensional case, and we provide this here when the Riemannian manifold is now time evolving in $\mathbb{R}^6$. Moreover, our earlier paper only dealt with inviscid materials. In this paper, we extend our results to viscous fluids, including the incompressible viscous fluids governed by the classical Navier-Stokes equations. In particular, we use the geometric theory to predict the critical profile initial data for the onset of turbulence in channel flow.

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