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Partially dissipative 2D Boussinesq equations with Navier type boundary conditions

Weiwei Hu [a], Yanzhen Wang [b], Jiahong Wu [a,*], Bei Xiao [a], Jia Yuan [c]

[a] Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, United States
[b] School of Mathematical Sciences, Xiamen University, Xiamen, 361005, China
[c] School of Mathematics and Systems Science, Beihang University, Beijing, 100191, China

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Dedicated to Professor Edriss S. Titi on the occasion of his sixtieth birthday

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ABSTRACT

This paper concerns itself with two systems of the 2D Boussinesq equations with partial dissipation in bounded domains with the Navier type boundary conditions. We attempt to achieve two main goals: first, to prove the global existence and uniqueness under minimal regularity assumptions on the initial data; and second, to provide a direct and transparent approach that explicitly reveals the impacts of the Navier boundary conditions. The 2D Boussinesq equations with partial dissipation have attracted considerable interests in the last few years, although most of the results are aimed at sufficiently regular solutions in the whole space or periodic domains. Larios et al. (2013) made serious efforts to minimize the regularity assumptions necessary for the uniqueness of solutions in the spatially periodic setting. In contrast to the whole space and the periodic domains, the Navier boundary conditions generate boundary terms and require compatibility conditions. In addition, due to the lack of boundary conditions for the pressure, we resort to the existence and regularity result on the associated Stokes problem with Navier boundary conditions. The uniqueness relies on the Yudovich techniques and the introduction of a lower regularity counterpart of the temperature.

1. Introduction

The Boussinesq equations model geophysical flows such as atmospheric fronts and oceanic circulation (see, e.g., [1–3]). In addition, they play an important role in the study of Rayleigh–Benard convection (see, e.g., [4,5]). This paper is concerned with two systems of partially dissipated 2D Boussinesq equations: the Boussinesq system with only kinematic dissipation (without thermal diffusion)

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \nu\,\Delta\mathbf{u} + \theta\mathbf{e}_2, \\ \partial_t\theta + \mathbf{u}\cdot\nabla\theta = 0, \\ \nabla\cdot\mathbf{u} = 0 \end{cases} \tag{1.1}$$

and its counterpart with only partial kinematic dissipation. Instead of the full Laplacian dissipation, this partially dissipative system has only vertical dissipation in the horizontal velocity equation and horizontal dissipation in the vertical velocity equation

$$\begin{cases} \partial_t u_1 + \mathbf{u}\cdot\nabla u_1 = -\partial_1 p + \nu\,\partial_{22}u_1, \\ \partial_t u_2 + \mathbf{u}\cdot\nabla u_2 = -\partial_2 p + \nu\,\partial_{11}u_2 + \theta, \\ \partial_t\theta + \mathbf{u}\cdot\nabla\theta = 0, \\ \nabla\cdot\mathbf{u} = 0. \end{cases} \tag{1.2}$$

In these equations $\mathbf{u} = \mathbf{u}(x,t)$ represents the 2D velocity with its horizontal and vertical components given by u_1 and u_2, respectively, $p = p(x,t)$ the pressure, $\theta = \theta(x,t)$ the temperature, $\mathbf{e}_2$ the unit vector in the vertical direction, and $\nu > 0$ represents the kinematic viscosity.

Our attention will be mainly focused on spatial domains $\Omega \subset \mathbb{R}^2$ that are bounded, connected and have smooth boundary, although the results presented here are also valid for $\Omega = \mathbb{R}^2$ and periodic domains, as explained later. We assume the velocity field $\mathbf{u}$ obeys the Navier boundary conditions. The Navier boundary conditions allow the fluid to slip along the boundary and require that the tangential component of the stress vector at the boundary be proportional to the tangential velocity. In the case of (1.1), the corresponding stress tensor $T = (T_{ij})$ is given by

$$T_{ij} = -\delta_{ij}p + 2\nu D_{ij}(\mathbf{u}), \qquad D_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j) \quad \text{or}$$

$$D(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)$$

* Corresponding author.
E-mail addresses: weiwei.hu@okstate.edu (W. Hu), 19020140154255@stu.xmu.edu.cn (Y. Wang), jiahong.wu@okstate.edu (J. Wu), bei.xiao@okstate.edu (B. Xiao), yuanjia@buaa.edu.cn (J. Yuan).

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and, if $\mathbf{n}$ and $\boldsymbol{\tau}$ are unit normal and tangent vectors to the boundary $\partial\Omega$, respectively, the proportionality is then represented by

$$\sum_{i,j=1,2} \tau_i T_{ij} n_j = \sigma \sum_{k=1,2} u_k \tau_k \quad \text{on} \quad \partial\Omega$$

for a constant σ. Due to the orthogonality of $\mathbf{n}$ and $\boldsymbol{\tau}$,

$$\sum_{i,j=1,2} \tau_i \delta_{ij} p\, n_j = 0.$$

The Navier boundary conditions for (1.1) then become

$$\mathbf{u}\cdot\mathbf{n} = 0, \qquad 2\,\mathbf{n}\cdot D(\mathbf{u})\cdot\boldsymbol{\tau} + \alpha\mathbf{u}\cdot\boldsymbol{\tau} = 0 \quad \text{on} \quad \partial\Omega, \tag{1.3}$$

where $\alpha > 0$ is a constant. For the system in (1.2), the kinematic dissipation is only partial. We follow the same principle to propose the corresponding Navier boundary condition. The corresponding stress tensor T associated with (1.2) is given by

$$T = -pI + 2\nu E(\mathbf{u})$$
$$\text{with} \quad E(\mathbf{u}) = \frac{1}{2}\begin{pmatrix} 0 & \partial_2 u_1 + \partial_1 u_2 \\ \partial_2 u_1 + \partial_1 u_2 & 0 \end{pmatrix}$$

and consequently, the Navier boundary conditions for (1.2) are

$$\mathbf{u}\cdot\mathbf{n} = 0, \qquad 2\,\mathbf{n}\cdot E(\mathbf{u})\cdot\boldsymbol{\tau} + \alpha\mathbf{u}\cdot\boldsymbol{\tau} = 0 \quad \text{on} \quad \partial\Omega. \tag{1.4}$$

As documented in many papers, the Navier boundary conditions are important in modeling many flows in the real world (see, e.g. [6–8]). Since the temperature is transported by the velocity field $\mathbf{u}$, no boundary condition should be imposed on θ.

A very important special case of the Navier boundary conditions in (1.3) or (1.4) is the stress-free boundary condition for which the vorticity $\omega = \nabla \times u$ vanishes on $\partial\Omega$,

$$\mathbf{u}\cdot\mathbf{n} = 0, \qquad \omega = \partial_1 u_2 - \partial_2 u_1 = 0 \quad \text{on} \quad \partial\Omega. \tag{1.5}$$

In addition, (1.1) and (1.2) will be supplemented with the initial data

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \theta(x,0) = \theta_0(x), \quad x \in \Omega. \tag{1.6}$$

The goal of this paper is three fold: first, to establish the global existence and uniqueness of solutions to (1.1) and (1.2) with their corresponding Navier boundary conditions, second, to obtain the uniqueness of solutions with minimal regularity assumption on the initial data (u_0, θ_0), and third, to employ a direct approach from which one can clearly see the impacts of the Navier boundary conditions as opposed to those of the periodic boundary conditions and of the whole space case. Our main results are stated in the following two theorems.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded and connected domain with sufficient smooth boundary, say $\partial\Omega \in C^{2,1}$ (Lipschitz continuous second derivatives). Let $\nu > 0$. Consider the initial and boundary value problem (IBVP) in* (1.1), (1.3) and (1.6) *with $\alpha > 0$ being a constant and*

$$\mathbf{u}_0 \in H^1(\Omega), \quad \nabla\cdot\mathbf{u}_0 = 0$$

and

$$\theta_0 \in L^2(\Omega)\cap L^\infty(\Omega), \qquad \int_\Omega \theta_0(x)\,dx = 0.$$

Then the IBVP (1.1), (1.3) and (1.6) *has a unique global (in time) strong solution $(\mathbf{u}, \theta)$ satisfying, for any $T > 0$,*

$$\mathbf{u} \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; \dot{H}^2(\Omega)),$$
$$\theta \in L^\infty(0,\infty; L^2(\Omega)\cap L^\infty(\Omega)),$$
$$\int_\Omega \theta(x,t)\,dx = 0 \ \textit{ for any } t \in [0,\infty). \tag{1.7}$$

When Ω is bounded, $\theta \in L^\infty(\Omega)$ automatically implies $\theta \in L^2(\Omega)$. We have kept $\theta \in L^2(\Omega)$ in the statement of Theorem 1.1 for the convenience of extension to the whole plane case below.

(1.2) involves only partial kinematic dissipation. The global well-posedness result obtained for this system is for the stress-free boundary condition

$$\mathbf{u}\cdot\mathbf{n} = 0, \qquad \partial_1 u_2 = \partial_2 u_1 = 0 \quad \text{on} \quad \partial\Omega. \tag{1.8}$$

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded and connected domain with sufficient smooth boundary, say $\partial\Omega \in C^{2,1}$. Let $\nu > 0$. Consider the initial and boundary value problem (IBVP) in* (1.2), (1.6) and (1.8) *with $\alpha > 0$ being a constant and*

$$\mathbf{u}_0 \in H^1(\Omega), \quad \nabla\cdot\mathbf{u}_0 = 0$$

and

$$\theta_0 \in L^2(\Omega)\cap L^\infty(\Omega), \qquad \int_\Omega \theta_0(x)\,dx = 0.$$

Then the IBVP (1.1), (1.6) and (1.8) *has a unique global (in time) strong solution $(\mathbf{u}, \theta)$ satisfying, for any $T > 0$,*

$$\mathbf{u} \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; \dot{H}^2(\Omega)),$$
$$\theta \in L^\infty(0,\infty; L^2(\Omega)\cap L^\infty(\Omega)),$$
$$\int_\Omega \theta(x,t)\,dx = 0 \ \textit{ for } t \in [0,\infty). \tag{1.9}$$

In contrast to the periodic boundary condition case or the whole space (with sufficient decay at ∞) case, the Navier type boundary conditions generate boundary terms and require compatibility conditions. In fact, the mean-zero assumption on θ_0 in Theorems 1.1 and 1.2, namely

$$\int_\Omega \theta_0(x)\,dx = 0 \tag{1.10}$$

is imposed to fulfill the compatibility condition in the proof of the uniqueness of the solutions. It is not difficult to understand that the results of Theorems 1.1 and 1.2 without (1.10) in the whole space or periodic domain case remain valid. More precisely, the following corollary (as consequences of the proofs of Theorems 1.1 and 1.2) holds.

Corollary 1.3. *Assume $\Omega = \mathbb{R}^2$ or $\Omega = [0, 2\pi]^2$ (periodic box). Assume $(\mathbf{u}_0, \theta_0)$ satisfies*

$$\mathbf{u}_0 \in H^1(\Omega), \quad \nabla\cdot\mathbf{u}_0 = 0, \qquad \theta_0 \in L^2(\Omega)\cap L^\infty(\Omega).$$

Then the initial value problem (IVP) (1.1) and (1.6) *or IVP* (1.2) and (1.6) *has a unique global strong solution $(\mathbf{u}, \theta)$ satisfying, for any $T > 0$,*

$$\mathbf{u} \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; \dot{H}^2(\Omega)),$$
$$\theta \in L^\infty(0,\infty; L^2(\Omega)\cap L^\infty(\Omega)).$$

The Navier–Stokes equations with Navier type boundary conditions have been studied extensively and there are many excellent references (see, e.g., [9–11]). The 2D Boussinesq equations with partial dissipation have recently attracted enormous attention, but most of the studies focus on the whole space or the periodic boundary. This paper is devoted to the partially dissipated Boussinesq equations with the Navier type boundary conditions. The theorems of this paper fill this gap. In addition, we strive to establish the uniqueness under minimal regularity assumptions on the initial data. [12] and [13] examined global solutions of (1.1) in the whole space $\mathbb{R}^2$ for $(\mathbf{u}_0, \theta_0) \in H^s(\mathbb{R}^2)$ with $s > 2$. [14] obtained the global existence and regularity of (1.2) in the whole space $\mathbb{R}^2$ for $(\mathbf{u}_0, \theta_0) \in H^3(\mathbb{R}^2)$. Larios, Lunasin and Titi [15] and Hu, Kukavica and Ziane [16] have made serious efforts to reduce the regularity