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Rigorous bounds on the effective thermal conductivity of composites with ellipsoidal inclusions

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A B S T R A C T

A new method is developed to derive the bounds of the effective thermal conductivity of composites with ellipsoidal inclusions. The transition layer for each ellipsoidal inclusion is introduced to make the trial temperature field for the upper bound and the trial heat flux field for the lower bound satisfy the continuous interface conditions which are absolutely necessary for the application of variational principles. According to the principles of minimum potential energy and minimum complementary energy, the bounds of the effective thermal conductivity of composites with ellipsoidal inclusions are rigorously derived. The effects of the distribution and geometric parameters of ellipsoidal inclusions on the bounds of the effective thermal conductivity of composites are analyzed. It should be shown that the present method is simple and needs not calculate the complex integrals of multi-point correlation functions. Meanwhile, the present method provides a powerful way to bound the effective thermal conductivity of composites, which can be developed to obtain a series of bounds by taking different trial temperature and heat flux fields. In addition, the present upper and lower bounds still are finite when the thermal conductivity of ellipsoidal inclusions tends to ∞ and 0, respectively.

1. Introduction

The problem of determining the effective physical properties of statistically homogeneous two-phase composites has an extensive history, cf. the reviews of Elsayed [5], Batchelor [1], Hashin [10] and Torquato [20]. Due to reasons of mathematical analogy, the results of the present paper can translate immediately into equivalent results for the effective electric conductivity, dielectric constant, and magnetic permeability of such composites.

Several methods have been developed to derive bounds on effective parameters of composites. Hashin [9] has derived bounds of the effective elastic moduli for statistically homogeneous and transversely isotropic materials by introducing polarization fields and applying variational principles. These bounds are expressed in terms of the volume fraction, which is the simplest statistical information related to the effective properties of two-phase composites. Willis [21] has derived the bounds of the effective elastic moduli for transversely isotropic composites consisting of matrix with aligned spheroidal particles or circular cracks. Other derivations and extensions have been given by Kantor and Bergman [13] using the analytic-function method and by Francfort and Murat [6] via the translation method. Just as pointed out by Hashin [10], it is necessary to incorporate additional geometrical information to improve the bounds.

In an early paper, Brown [3] has demonstrated the futility of trying to determine exact effective properties of two-phase composites, such as the effective thermal conductivity, without a complete knowledge of the statistical information about

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composites. Without this knowledge, however, bounds on effective properties can be derived by variational principles. Hori [11,12] has obtained the bounds for the effective conductivity of macroscopically anisotropic media by perturbation expansions. However, the n-point microstructural parameters involve the derivatives of the correlation functions rather than the correlation functions themselves. Sen and Torquato [19] have derived a new perturbation expansion for the effective conductivity of d-dimensional two-phase media of arbitrary topology which depends on n-point parameters. Phan-Thien and Milton [18] have derived the third- and fourth-order bounds for the effective thermal conductivity of composites in terms of the perturbation solution to the effective thermal conductivity problem for an N-component material. Bruno [4] has investigated the effective conductivity of two-phase composites and given the corresponding bounds by using the complex variable method. More restrictive bounds on the effective properties which include additional information about the microstructure of composites have also been obtained by Beran and Molyneux [2], McCoy [16], Milton and Phan-Thien [15], and Pham [17].

In the present paper, we seek to give the rigorous bounds on the effective thermal conductivity of composites with ellipsoidal inclusions. In Section 2, some basic formulas and conditions which are used to derive the bounds of the effective thermal conductivity of composites are given. In Section 3, the transition layers between the remnant matrix and ellipsoidal inclusions are established to make the trial temperature field in the composite keep continuous. According to the principle of minimum potential energy, the upper bound on the effective thermal conductivity of composites is derived. In Section 4, the trial heat flux field in the composite is constructed to satisfy the differential equation, boundary condition and interface conditions between transition layers and other regions (ellipsoidal inclusions and remnant matrix). The lower bound on the effective thermal conductivity of composites is derived in terms of the principle of minimum complementary energy. In Section 5, the upper and lower bounds on the effective thermal conductivity are calculated. The effects of the distribution and geometric parameters of ellipsoidal inclusions on the bounds of the effective thermal conductivity are analyzed. It needs to be pointed out that the present method has the following advantages: (1) it is simple; (2) it does not need to calculate the complex integrals of multi-point correlation functions; (3) it provides a powerful way to bound the effective thermal conductivity of composites with ellipsoidal inclusions; (4) it can consider the limiting cases that the thermal conductivity of ellipsoidal inclusions tends to ∞ and 0, respectively. Finally, some conclusions are summarized in Section 6.

In what follows, the summation convention is used only for subscripts with Small Latin letters, and Great Latin and Greek letters in superscripts and subscripts are not summed. The subscript with a bar is called as the dummy suffix which is not summed and only takes the same value with the non-dummy suffix. A comma denotes partial differentiation, for example, $f_{,i}(\mathbf{x})$ means $\partial f(\mathbf{x})/\partial x_i$.

2. Basic relations

Consider a statistically homogeneous two-phase composite with the representative volume V where the volumes of ellipsoidal inclusions and matrix are V_I and V_M, respectively, and boundary surface S. Let the thermal conductivities of ellipsoidal inclusions and matrix be denoted by σ_I and σ_M, respectively. The equations governing steady-state heat conduction at point $\mathbf{x} \in V$ are as follows:

Differential equation:

$$Q^K_{i,i}(\mathbf{x}) = 0, \quad (K = I, M) \tag{1}$$

where $Q^I_i(\mathbf{x})$ and $Q^M_i(\mathbf{x})$ are the components of heat flux vectors along the x_i-direction in ellipsoidal inclusions and matrix, respectively.

Constitutive relation:

$$Q^K_i(\mathbf{x}) = \sigma_K E^K_i(\mathbf{x}) \quad (K = I, M) \tag{2}$$

where $E^I_i(\mathbf{x})$ and $E^M_i(\mathbf{x})$ are the components of temperature gradient vectors along the x_i-direction in ellipsoidal inclusions and matrix, respectively.

Gradient equation:

$$E^K_i(\mathbf{x}) = -T^K_{,i}(\mathbf{x}), \quad (K = I, M) \tag{3}$$

where $T^I(\mathbf{x})$ and $T^M(\mathbf{x})$ are the temperature fields in ellipsoidal inclusions and matrix, respectively.

Homogeneous boundary conditions:

$$T(S) = -E^0_i x_i, \quad \mathbf{x} \in S \tag{4a}$$

$$Q_n(S) = Q^0_i n_i, \tag{4b}$$

where E^0_i and Q^0_i are uniform temperature gradient and heat flux components, respectively, $Q_n(S)$ is the normal component of the heat flux vector on the boundary, x_i are Cartesian coordinates, and n_i are the components of the unit normal vector on surface S. Eqs. (4a) and (4b) are called as the first and second kinds of boundary conditions, respectively.