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On the Lagrangian and Eulerian analyticity for the Euler equations

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Dedicated to Professor Edriss S. Titi on the occasion of his sixtieth birthday

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ABSTRACT

We address preservation of the Lagrangian analyticity radius of solutions to the Euler equations belonging to natural analytic space based on the size of Taylor (or Gevrey) coefficients. We prove that if the solution belongs to such space, then the solution also belongs to it for a positive amount of time. We also prove the local analog of this result for a sufficiently large Gevrey parameter; however, we show that the preservation holds independently of the size of the radius. Finally, we construct a solution which shows that the first result does not hold in the Eulerian setting.

1. Introduction

The motion of a fluid can be described either in reference to a fixed space–time point (x, t) (Eulerian coordinates) or by following the flow lines of individual fluid particles (Lagrangian coordinates). It is known that these two formulations are significantly different. There are examples of 3D steady flows with complicated particle paths where streamlines have the space filling property (cf. [1]). Conversely, there are cases where the Lagrangian formulation is better adapted.

In this paper we study the Lagrangian formulation of the incompressible Euler equations on $\mathbb{R}^d$, where $d \in \{2, 3\}$. It is known from [2–5] that a solution of the Euler equations remains analytic (or more generally Gevrey regular) if it is so initially, as long as the solution exists. It is an interesting question whether the solution actually belongs to the same analytic space for a positive amount of time. This is since while solutions remain analytic (or more generally Gevrey regular), the radius may actually decrease. In [6], it was shown that the Lagrangian solution of the Euler equation (or more generally Gevrey) belongs to the same analytic space as the initial data for a positive amount of time. The space in [6] requires summability of Taylor coefficients. Moreover, it was shown, that the same statement does not hold in the Eulerian setting, i.e., a solution was constructed whose sum of Taylor coefficients strictly decreases as time increases.

In this paper, we address the issue of persistence with respect to the analytic space which is however based on the natural supremum condition for the Taylor coefficient (i.e., uniform analyticity/Gevrey regularity) as studied say in [7] and many other works (cf. [8–12]). More precisely, by defining a suitable Lagrangian Gevrey-class norm, we prove that if the initial velocity gradient is of Gevrey-class s, where $s \geq 1$, then the Sobolev solution $v(\cdot, t) \in C([0, T]; H^r(\mathbb{R}^d))$ is of Gevrey-class s for all $t < T$. We would like to emphasize that the main result in [6] and Theorem 3.1 here do not imply each other; they both establish that the solution persists in a certain analytic space, which have different definitions (and are both natural in a certain sense).

In the second main result, Theorem 4.1, we establish a local version of the preservation of the Lagrangian radius. Namely, we prove that the local Gevrey radius is preserved on a positive time interval, regardless of the size of the Gevrey radius (i.e., even if it is larger than the diameter of the domain). The statement requires the usual assumption that the Sobolev norm of the velocity stays finite for a small time, i.e., $\sup_{t\in[0,T]} \|v(\cdot, t)\|_{H^r(B(0,R))} < \infty$, and that the Gevrey parameter is sufficiently large. It would be interesting to prove this for the full range of Gevrey radii, although this may be true in the analytic case only for radii sufficiently small. In a third main result, Theorem 5.1, we demonstrate that our global analyticity result does not hold in the Eulerian coordinates by constructing an explicit initial data for which the radius strictly decreases. A similar example was provided in [6]; however, the nature of our space (the supremum assumption instead of integrability), allows for a simpler construction, which can be carried out in the original variables (rather than in the Fourier space as in [6]).

The Lagrangian approach has gained a strong interest due to the possibility that the Lagrangian paths could be analytic in time, a

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fact which was first observed by Serfati [13]. Earlier, Chemin [14] proved that the Lagrangian trajectories are C^∞ smooth. More recently, in [15–17], the authors developed an elementary theory of analytic fluid particle trajectories. Their work is based on Cauchy's long-forgotten manuscript (1815) on Lagrangian formulation of 3D incompressible Euler equations in terms of Lagrangian invariants. For further results on time-analyticity of Lagrangian trajectories, we refer the reader to [18–20] and to [21] for advantages of the Lagrangian formulation in the Hölder class.

The remarkable difference in time analyticity of the two formulations suggests a further discussion on the analytic regularity of solutions. There is quite a rich history on the persistence of real-analyticity of the solutions in both two and three dimensions with many works pointing out the contrast between the Lagrangian and Eulerian analytic regularity (cf. [2,3,22–26]). In [3], the authors established a lower bound of the form $\exp(-C\exp(CT))/C$ for the radius of analyticity in 2D, with the constant C depending on the initial data. Finding explicit rates for the decay of the radius of analyticity was studied further using different methods in [27,28] for the interior and in [29] for the boundary value problem. Also, in [30,31] the authors extended the results in [28,29] to the non-analytic Gevrey classes and improved the dependence on the initial datum. For space-periodic domains, Levermore and Oliver gave in [5] a proof of persistence of analyticity for all times. Their proof is based on a characterization of Gevrey classes in terms of decay of Fourier coefficients.

Furthermore, the shear flow example (cf. [32,33]) has generated numerous constructions of explicit solutions to periodic 3D Euler equations whose radius of analyticity decays for all time. In the analytic class (Gevrey-1 class) case, cf. [30, Remark 1.3] and [6, Theorem 1.3]. Also, one can construct an example in the non-analytic Gevrey classes with $s > 1$ (cf. [34]). Moreover, in [6] the authors point out the difference of behaviors in the two formulations; the radius of analyticity is conserved locally in time for the Lagrangian formulation, whereas it deteriorates instantaneously in the Eulerian one. A similar contrast is also observed in terms of solvability in anisotropic classes. The Lagrangian formulation is locally well-posed in anisotropic classes yet the equations are ill-posed in Eulerian coordinates.

The paper is organized as follows. In Section 2, we introduce the Gevrey-class space using the supremum over Taylor coefficients (as opposed to their sum as in [6]). Furthermore, the well-posedness of Lagrangian formulation in anisotropic Gevrey classes holds for local solutions as well. In Section 3, we prove that if the Gevrey regularity parameter is sufficiently large (cf. (4.1) below), then the analyticity radius is preserved in the future, regardless of its size. It is an interesting question if this theorem can be extended for analytic class as well. Finally, in Section 5, we provide a counterexample to Theorem 3.1 in the Eulerian setting.

The paper is dedicated to Professor Edriss Titi on the occasion of his sixtieth birthday in admiration of his work and in appreciation for his support throughout the years.

2. Euler equations in Lagrangian coordinates

The incompressible homogeneous Euler equations in $\mathbb{R}^d$, for $d = 2, 3$, are given by the system of equations

$$u_t + u\cdot\nabla u + \nabla p = 0 \tag{2.1}$$
$$\nabla\cdot u = 0 \tag{2.2}$$
$$u(x,0) = u_0(x) \tag{2.3}$$

for $(x,t) \in \mathbb{R}^d \times [0,\infty)$. The above system models the flow of an incompressible, homogeneous, and inviscid fluid, where $u(x,t) = (u^1,\ldots,u^d)$ denotes the fluid velocity and $p(x,t)$ the pressure.

We rewrite the Euler equations using the particle trajectory mapping $X(\cdot,t) : \alpha \mapsto X(\alpha,t) \in \mathbb{R}^d$, where $t \geq 0$. The vector $X(\alpha,t) = (X_1,\ldots,X_d)$ denotes the location of a fluid particle at time t that is initially placed at the Lagrangian label α, and is given by an ODE

$$\partial_t X(\alpha,t) = u(X(\alpha,t),t) \tag{2.4}$$
$$X(\alpha,0) = \alpha. \tag{2.5}$$

Composing the velocity and the pressure with X, we obtain the Lagrangian velocity v and the pressure q by

$$v(\alpha,t) = u(X(\alpha,t),t)$$
$$q(\alpha,t) = p(X(\alpha,t),t).$$

Also, denote by Y_i^k the (k,i)th entry of the inverse of the Jacobian of X, i.e.,

$$Y(\alpha,t) = (\nabla_\alpha X(\alpha,t))^{-1}.$$

We then write the Lagrangian formulation of the Euler equations as

$$\partial_t v^i + Y_i^k \partial_k q = 0, \qquad i = 1,\ldots,d \tag{2.6}$$
$$Y_i^k \partial_k v^i = 0 \tag{2.7}$$

with the summation convention on repeated indices understood. The system (2.6)–(2.7) is supplemented with the initial conditions

$$v(\alpha,0) = v_0(\alpha) = u_0(\alpha)$$
$$Y(\alpha,0) = I.$$

Differentiating (2.4) with respect to the Lagrangian labels along with using $\det(\nabla X) = 1$ and inverting the matrix in the resulting equation, we get

$$Y_t = -Y : (\nabla v) : Y \tag{2.8}$$

where the symbol : denotes the matrix multiplication. Taking the curl of the Eq. (2.1) and using $\nabla \times (u\cdot\nabla u) = u\cdot\nabla\omega - \omega\cdot\nabla u$, we obtain

$$\partial_t \omega + u\cdot\nabla\omega = \omega\cdot\nabla u.$$

Hence,

$$\frac{D\omega}{Dt} = \omega\cdot\nabla u$$

where D/Dt is the convective derivative, i.e., the derivative along the particle trajectories. In 2D flows the vorticity is conserved, i.e., $\zeta(\alpha,t) = \omega_0(a)$ for $t \geq 0$. Denoting the sign of the permutation $(1,2) \mapsto (i,j)$ by ϵ_{ij}, we may write the Euler system as

$$\epsilon_{ij} Y_i^k \partial_k v^j = Y_1^k \partial_k v^2 - Y_2^k \partial_k v^1 = \omega_0 \tag{2.9}$$
$$Y_i^k \partial_k v^i = Y_1^k \partial_k v^1 + Y_2^k \partial_k v^2 = 0. \tag{2.10}$$

If $d = 3$, we may use the vorticity-transport formula $\zeta^i(\alpha,t) = \partial_k X^i(\alpha,t)\omega_0^k(\alpha)$ and proceed as in [6] to write the Euler equations as

$$\epsilon_{ijk} Y_i^m Y_j^l \partial_l v^k = Y_i^m \zeta^i = \omega_0^m, \qquad m = 1,2,3 \tag{2.11}$$
$$Y_i^k \partial_k v^i = 0. \tag{2.12}$$

The Eq. (2.11), derived in [6], represents a way to write the Cauchy invariance formula without involving X.

3. The preservation of the Gevrey radius

We start by recalling the definition of Gevrey spaces. For any $s \geq 1$, we define the s-Gevrey norm with radius $\delta > 0$ by

$$\|f\|_{G_{s,\delta}} = \sup_{|\alpha|\geq 0} \frac{\delta^{|\alpha|}}{|\alpha|!^s} \|\partial^\alpha f\|_{H^r}$$