ARTICLE IN PRESS

Physica D ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Wave breaking for the Stochastic Camassa–Holm equation☆

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HIGHLIGHTS

- The stochastic Camassa–Holm (SCH) equation is derived variationally.
- Peakon solutions and isospectrality conditions are found for the SCH equation.
- Wave breaking also survives introducing stochasticity into the SCH equation.

ARTICLE INFO

Article history:
Available online xxxx

Keywords:
Emergent singularities
Stochastic PDEs
Nonlinear waves

ABSTRACT

We show that wave breaking occurs with positive probability for the Stochastic Camassa–Holm (SCH) equation. This means that temporal stochasticity in the diffeomorphic flow map for SCH does not prevent the wave breaking process which leads to the formation of peakon solutions. We conjecture that the time-asymptotic solutions of SCH will consist of emergent wave trains of peakons moving along stochastic space–time paths.

1. The deterministic Camassa–Holm (CH) equation

The deterministic CH equation, derived in [1], is a nonlinear shallow water wave equation for a fluid velocity solution whose profile $u(x,t)$ and its gradient both decay to zero at spatial infinity, $|x|\to\infty$, on the real line $\mathbb{R}$. Namely,

$$u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}\,, \tag{1.1}$$

where subscripts t (resp. x) denote partial derivatives in time (resp. space). This nonlinear, nonlocal, completely integrable PDE may be written in *Hamiltonian form* for a momentum density $m := u - u_{xx}$ undergoing coadjoint motion, as [1]

$$m_t = \{m, h(m)\} = -(\partial_x m + m\partial_x)\frac{\delta h}{\delta m}\,, \tag{1.2}$$

which is generated by the Lie–Poisson bracket

$$\{f,h\}(m) = -\int \frac{\delta f}{\delta m}(\partial_x m + m\partial_x)\frac{\delta h}{\delta m}\,dx \tag{1.3}$$

and Hamiltonian function

$$h(m) = \frac{1}{2}\int_{\mathbb{R}} mK*m\,dx = \frac{1}{2}\int_{\mathbb{R}} u^2 + u_x^2\,dx$$

$$= \frac{1}{2}\|u\|^2_{H^1} = const. \tag{1.4}$$

Here, $K*m := \int K(x,y)m(y,t)dy$ denotes convolution of the momentum density m with Green's function of the Helmholtz operator $L = 1-\partial_x^2$, so that

$$\frac{\delta h}{\delta m} = K*m = u \quad \text{with} \quad K(x-y) = \frac{1}{2}\exp(-|x-y|)\,. \tag{1.5}$$

Alternatively, the CH equation (1.1) may be written in advective form as

$$u_t + uu_x = -\,\partial_x\Big(K*\big(u^2+\frac{1}{2}u_x^2\big)\Big)$$

$$= -\,\partial_x\int_{\mathbb{R}}\frac{1}{2}\exp(-|x-y|)\left(u^2(y,t)+\frac{1}{2}u_y^2(y,t)\right)dy\,. \tag{1.6}$$

The deterministic CH equation admits signature solutions representing a wave train of peaked solitons, called *peakons*, given by

$$u(x,t) = \frac{1}{2}\sum_{a=1}^{M} p_a(t)e^{-|x-q_a(t)|} = K*m\,, \tag{1.7}$$

which emerge from smooth confined initial conditions for the velocity profile. Such a sum is an *exact solution* of the CH equation (1.1) provided the time-dependent parameters $\{p_a\}$ and $\{q_a\}$, $a = 1,\ldots,M$, satisfy certain canonical Hamiltonian equations, to be discussed later. In fact, the peakon velocity wave train in (1.7) is the *asymptotic solution* of the CH equation for any spatially confined C^1 initial condition, $u(x,0)$.

☆ Paper submitted to the Physica D special issue *Nonlinear Partial Differential Equations in Mathematical Fluid Dynamics* dedicated to Prof. Edriss S. Titi on the occasion of his 60th birthday. Work partially supported by the EPSRC Standard Grant EP/N023781/1.

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https://doi.org/10.1016/j.physd.2018.02.004

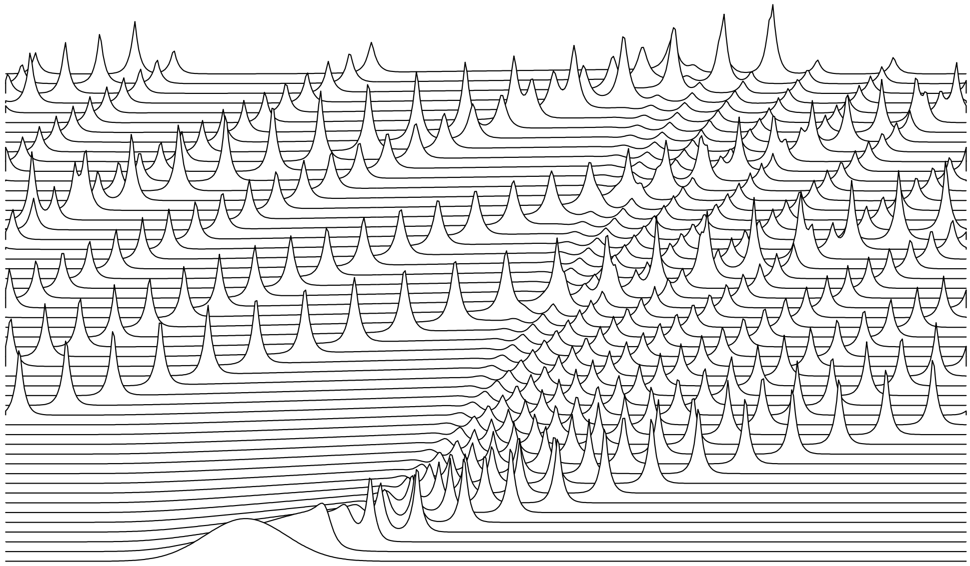

Fig. 1.1. Under the evolution of the CH equation (1.1), an ordered *wave train of peakons* emerges from a smooth localized initial condition (a Gaussian). The speeds are proportional to the heights of the peaks. The spatial profiles of the velocity at successive times are offset in the vertical to show the evolution. The peakon wave train eventually wraps around the periodic domain, thereby allowing the faster peakons which emerge earlier to overtake slower peakons emerging later from behind in collisions that conserve momentum and preserve the peakon shape but cause phase shifts in the positions of the peaks, as discussed in [1].

Remark 1. The peakon-train solutions of CH represent an *emergent phenomenon*. A wave train of peakons emerges in solving the initial-value problem for the CH equation (1.1) for any smooth spatially confined initial condition. An example of the emergence of a wave train of peakons from a Gaussian initial condition is shown in Fig. 1.1.

Remark 2. By Eq. (1.5), the momentum density corresponding to the peakon wave train (1.7) in velocity is given by a sum over delta functions in momentum density, representing the *singular solution*,

$$m(x,t) = \sum_{a=1}^{M} p_a(t)\,\delta(x - q_a(t))\,, \tag{1.8}$$

in which the *delta function* $\delta(x-q)$ is defined by

$$f(q) = \int f(x)\delta(x-q)\,\mathrm{d}x\,, \tag{1.9}$$

for an arbitrary smooth function f. Physically, the relationship (1.8) represents the dynamical coalescence of the CH momentum density into particle-like coherent structures (Young measures) which undergo elastic collisions as a result of their nonlinear interactions. Mathematically, the singular solutions of CH are captured by recognizing that the singular solution ansatz (1.8) itself is an equivariant momentum map from the canonical phase space of M points embedded on the real line, to the dual of the vector fields on the real line. Namely,

$$m : T^*\mathrm{Emb}(\mathbb{Z},\mathbb{R}) \to \mathfrak{X}(\mathbb{R})^*. \tag{1.10}$$

This momentum map property explains, for example, why the singular solutions (1.8) form an invariant manifold for any value of M and why their dynamics form a canonical Hamiltonian system [2].

The complete integrability of the CH equation as a Hamiltonian system follows from its isospectral problem.

Theorem 3 (*Isospectral Problem for CH* [1]). *The CH equation in (1.1) follows from the compatibility conditions for the following CH isospectral eigenvalue problem and evolution equation for the real eigenfunction $\psi(x,t)$,*

$$\psi_{xx} = \left(\frac{1}{4} - \frac{m}{2\lambda}\right)\psi\,, \tag{1.11}$$

$$\partial_t\psi = -(\lambda + u)\psi_x + \frac{1}{2}u_x\psi\,, \tag{1.12}$$

with real isospectral parameter, λ.

Proof. By direct calculation, equating cross derivatives $\partial_t\psi_{xx} = \partial_x^2\partial_t\psi$ using Eqs. (1.11) and (1.12) implies the CH equation in (1.1), provided $d\lambda/dt = 0$. □

Remark 4. The complete integrability of the CH equation as a Hamiltonian system and its soliton paradigm explain the emergence of peakons in the CH dynamics. Namely, their emergence reveals the initial condition's soliton (peakon) content.

1.1. Steepening lemma: the mechanism for peakon formation

In the following we will continue working on the entire real line $\mathbb{R}$, although similar results are also available for a periodic domain with only minimal effort. We use the notation $\|u\|_2$, $\|u\|_{1,2}$ and $\|u\|_\infty$ to denote, respectively,

$$\|u\|_2^2 := \int_{-\infty}^{\infty}(u^2)\,\mathrm{d}y\,,\quad \|u\|_{1,2}^2 := \int_{-\infty}^{\infty}\left(u^2 + \frac{1}{2}u_y^2\right)\mathrm{d}y\,,\quad \text{and}$$
$$\|u\|_\infty := \sup_{x\in\mathbb{R}}\|u(x)\|.$$

Remark 5 (*Local Well-Posedness of CH*). As reviewed in [2], the deterministic CH equation (1.1) is locally well posed on $\mathbb{R}$, for initial conditions in H^s with $s > 3/2$. In particular, with such initial data, CH solutions are C^∞ in time and the Hamiltonian $h(m)$ in (1.4) is bounded for all time,

$$h := \|u(\cdot,t)\|_{1,2} < \infty.$$

In fact, CH solutions preserve the Hamiltonian in (1.4) given by the $\|u(\cdot,t)\|_{1,2}$ norm

$$\|u(\cdot,t)\|_{1,2} = h = \mathit{constant}\,,\quad \text{for all } x\in\mathbb{R}. \tag{1.13}$$

By a standard Sobolev embedding theorem, (1.13) also implies the useful relation that

$$M := \sup_{t\in[0,\infty)}\|u(\cdot,t)\|_\infty < \infty. \tag{1.14}$$

Please cite this article in press as: D. Crisan, D.D. Holm, Wave breaking for the Stochastic Camassa–Holm equation, Physica D (2018), https://doi.org/10.1016/j.physd.2018.02.004.