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Turbulence in vertically averaged convection

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HIGHLIGHTS

- Vertical average of 3D Rayleigh–Bénard results in 2D Navier–Stokes system.
- Body force for this system which depends on 3D velocity is estimated.
- Dissipation wave number is estimated in terms of Grashof numbers.
- One side of the dissipation law holds up to a shape factor of the force.

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This paper is dedicated to Edriss Titi for all he has done as a mathematician and a friend

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ABSTRACT

The vertically averaged velocity of the 3D Rayleigh–Bénard problem is analyzed and numerically simulated. This vertically averaged velocity satisfies a 2D incompressible Navier–Stokes system with a body force involving the 3D velocity. A time average of this force is estimated through time averages of the 3D velocity. Relations similar to those from 2D turbulence are then derived. Direct numerical simulation of the 3D Rayleigh–Bé nard is carried out to test how prominent the features of 2D turbulence are for this Navier–Stokes system.

1. Introduction

There are fundamental differences between the statistical theories of isotropic, homogeneous turbulence in three and two space dimensions. The 1941 Kolmogorov–Obukhov theory [1,2] in 3D has in the inertial range a $\kappa^{-5/3}$ energy spectrum and direct cascade of energy toward smaller scales, while in the Batchelor–Kraichnan–Leith theory [3–5] for 2D the spectrum in the inertial range is found to scale as κ^{-3} (up to a log-correction) with the direct cascade being one of enstrophy. Moreover at large length scales in 2D there is expected to be an inverse cascade of energy associated with again, a $\kappa^{-5/3}$ energy spectrum. The direct cascade means that the energy in 3D (resp., enstrophy in 2D) is, on average, transferred to smaller scales at a rate which is comparable to the energy injection rate ϵ (resp. enstrophy injection rate η) down to scales small enough to be dissipated by viscous effects. Dimensional analysis associates this dissipation length scale with the wave numbers $\kappa_\epsilon = (\epsilon/\nu^3)^{1/4}$ in 3D and $\kappa_\eta = (\eta/\nu^3)^{1/6}$ in 2D. These pictures of turbulence are generally made under the assumption that energy is injected over a finite range of wave numbers, say $[\underline{\kappa}, \overline{\kappa}]$. In that case, the condition $\overline{\kappa} \ll \kappa_\sigma = (\eta/\epsilon)^{1/2}$ is sufficient for an enstrophy cascade in 2D [6], and in fact necessary [7].

For turbulence modeled by the Navier–Stokes equations (NSE) on a torus in $\mathbb{R}^d$, the energy is injected through an external body force $\mathbf{f}$. If the force is constant in time, it is convenient to gauge turbulent behavior by the dimensionless Grashof number, $G = \|\mathbf{f}\|_{L^2}/(\nu^2\kappa_0^d)$, where ν is the kinematic viscosity and κ_0 the smallest wavenumber. The extent of the inertial range can be controlled through upper and lower bounds on κ_ϵ and κ_η in terms of G, which can be specified by the body force. To meet the condition $\overline{\kappa} \ll \kappa_\sigma$ for an enstrophy cascade, however, requires knowledge of the solution. It is not clear what type of finite mode force would produce a solution meeting that condition: how its Fourier coefficients are distributed, how they should depend on time. A common approach in direct numerical simulations of turbulence is to take the force to be stochastic in a finite number of arbitrarily chosen modes. By considering instead a force which incorporates effects from a 3D boundary layer, we expect to see turbulent behavior that has a natural connection to a physical problem. This is the main motivation for what follows.

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In this paper we study the 2D NSE which results upon taking the vertical average of the momentum equation in the 3D Rayleigh–Bénard system. This 2D system has a time-dependent force $\mathbf{f}_2$ comprised of a Reynolds stress (difference between the nonlinear term of the averages and the average of the nonlinear term) as well as a boundary shear contribution (average of the vertical diffusion). Since both of these forcing terms depend on the 3D velocity of the Rayleigh–Bénard problem, they are time dependent, and expected to involve all modes. We deal with the time dependence of the force by considering two Grashof-type numbers $G_* = \|\langle \mathbf{f}_2\rangle\|_{L^2}/(\nu\kappa_0)^2$ and $G^* = \langle \|\mathbf{f}_2\|_{L^2}^2\rangle^{1/2}/(\nu\kappa_0)^2$, where $\langle\cdot\rangle$ is an extended time average (see (3.14)). This approach differs from another for 2D turbulence with time dependent forcing in all scales in [8] where bounds are found in terms of ess $\sup_t \|f(t)\|L^2/(\nu\kappa_0)^2$.

We derive in Section 2 the Reynolds stress and boundary shear terms that force the vertically averaged momentum equation. The resulting 2D Navier–Stokes as well as the 3D Rayleigh–Bénard system are both expressed in functional form in Section 3 along with essential preliminary material. We recover in Section 4 an identity from [9] expressing the time average of enstrophy for the 3D velocity in terms of the Rayleigh and Nusselt numbers, Ra and Nu. Then, for any solution to the Rayleigh–Bénard system satisfying a maximum principle for the temperature, we derive bounds on the Reynolds numbers for the 3D and vertically averaged velocity. In Section 5 we prove relations akin to some for 2D turbulence. This starts with upper and lower bounds on κ_η in terms of G^* and G_* respectively. Since the body force is expected to contribute to the enstrophy at all wave numbers, it can be interpreted as a source external to any range. This is why we include it in a pseudo-flux function of enstrophy (per unit mass), which is comparable to the enstrophy dissipation rate over a predictable range of scales. One side of the dissipation law is shown to hold up to a shape factor of the force $\mathbf{f}_2$. In Section 6 we derive an upper bound on the time average of $\|\mathbf{f}_2\|_{L^2}^2$.

Direct numerical simulations are presented in Section 7. There we compute the solution to the 3D Rayleigh–Bénard problem in order to calculate the vertically averaged velocity as well as Reynolds stress and boundary shear forces. The energy spectra at several Rayleigh numbers are compared. At $\mathrm{Ra} = 5\times10^5$ the spectrum shows a κ^{-3} range, while at $\mathrm{Ra} = 1\times10^6, 2.5\times10^6$, the spectra display a $\kappa^{-5/3}$. The bounding expressions for the energy dissipation wave number are plotted from $\mathrm{Ra} = 5\times10^4$ to $\mathrm{Ra} = 2.5\times10^6$, and indicate that the lower bound may be considerably sharper than the upper bound. The computations are done using a pseudospectral code with 512 ×512 ×16 Fourier–Chebyshev modes, before dealiasing. Due to the low resolution in the vertical direction, the results of these simulations are to be taken as suggestive.

2. Vertically averaged velocity

The Boussinesq approximation of convection between two plates can be written as

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = g\alpha\delta_\theta\theta\mathbf{e}_3, \tag{2.1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2.2}$$

$$\theta_t + (\mathbf{u}\cdot\nabla)\theta - \beta\Delta\theta = 0 \tag{2.3}$$

in the region

$$\Omega = \{(x,y,z) : (x,y)\in(0,L)^2, 0<z<h\},$$

with boundary conditions

$$\mathbf{u} = 0 \text{ at } z=0 \text{ and } z=h, \tag{2.4}$$

$$p, \mathbf{u}, \theta \text{ periodic in horizontal directions } x, y, \tag{2.5}$$

$$\theta(0) = 1, \quad \theta(h) = 0\,. \tag{2.6}$$

Here, δ_θ is the physical positive temperature difference between the bottom and top plates. We work with a modified pressure $p = \tilde{p}/\rho_0 + gz$, where $\tilde{p}$ is the physical pressure, ρ_0 the density and g the magnitude of a constant gravitational field. We denote the heat conduction coefficient by β, the volume expansion coefficient by α, and the standard unit vector in the z-direction by $\mathbf{e}_3$. The components of the velocity in three dimensions are denoted as $\mathbf{u} = (u, v, w)$.

We consider the vertical average of the horizontal components of the 3D momentum equation

$$\bar{\mathbf{u}}_t - \nu\Delta\bar{\mathbf{u}} + (\bar{\mathbf{u}}\cdot\nabla)\bar{\mathbf{u}} + \nabla\bar{p} = \mathbf{F}^\partial(\mathbf{u}) + \mathbf{F}^{\mathrm{R}}(\mathbf{u})\,, \tag{2.7}$$

in $\overline{\Omega} = (0,L)^2$, where $\bar{\mathbf{u}} = (\bar{u}, \bar{v})$,

$$\bar{u} = \bar{u}(x,y) = \frac{1}{h}\int_0^h u(x,y,z)\,dz\,, \quad \text{similarly for } \bar{v}, \bar{p}$$

and the balancing force consists of a boundary shear

$$\mathbf{F}^\partial = \nu\overline{\mathbf{u}_{zz}} = \frac{\nu}{h}\begin{pmatrix} u_z(x,y,h) - u_z(x,y,0) \\ v_z(x,y,h) - v_z(x,y,0)\end{pmatrix}$$

and Reynolds stress

$$\mathbf{F}^{\mathrm{R}} = \begin{pmatrix} F_1^{\mathrm{R}} \\ F_2^{\mathrm{R}}\end{pmatrix} = \begin{pmatrix} \bar{u}\,\bar{u}_x + \bar{v}\,\bar{u}_y - \overline{uu_x + vu_y + wu_z} \\ \bar{u}\,\bar{v}_x + \bar{v}\,\bar{v}_y - \overline{uv_x + vv_y + wv_z}\end{pmatrix}.$$

From the incompressibility of the 3D velocity and the boundary condition $\mathbf{u} = 0$ at the top and bottom, we have that the 2D velocity $\bar{\mathbf{u}}$ is divergence-free:

$$0 = \frac{1}{h}\int_0^h u_x + v_y + w_z\,dz = \bar{u}_x + \bar{v}_y + \frac{1}{h}[w(h) - w(0)] = \bar{u}_x + \bar{v}_y\,.$$

We now rewrite the Reynolds stress in a more convenient form. First, we integrate by parts, and use 3D incompressibility to obtain

$$\overline{wu_z} = -\overline{uw_z} = \overline{u(u_x + v_y)}. \tag{2.8}$$

Then using (2.8) in the first component of the Reynolds stress, and applying 2D incompressibility $\bar{u}_x = -\bar{v}_y$, we have

$$\begin{aligned} F_1^{\mathrm{R}} &= \bar{u}\,\bar{u}_x + \bar{v}\,\bar{u}_y - \overline{uu_x + vu_y + wu_z} \\ &= \bar{u}\,\bar{u}_x + \bar{v}\,\bar{u}_y - 2\overline{uu_x} - \overline{vu_y} - \overline{uv_y} \\ &= 2\bar{u}\,\bar{u}_x + \bar{v}\,\bar{u}_y + \bar{u}\,\bar{v}_y - 2\overline{uu_x} - \overline{vu_y} - \overline{uv_y}\,. \end{aligned}$$

These six final terms can be regrouped through

$$\begin{aligned} \overline{[(u-\bar{u})^2]_x} &= 2\overline{(u-\bar{u})(u_x - \bar{u}_x)} = 2\,(\overline{uu_x} - \bar{u}\,\bar{u}_x - \bar{u}\,\bar{u}_x + \bar{u}\,\bar{u}_x) \\ &= 2\overline{uu_x} - 2\bar{u}\,\bar{u}_x \end{aligned}$$

$$\begin{aligned} \overline{[(u-\bar{u})(v-\bar{v})]_y} &= \overline{(u_y - \bar{u}_y)(v-\bar{v})} + \overline{(u-\bar{u})(v_y - \bar{v}_y)} \\ &= \overline{vu_y} - \bar{v}\,\bar{u}_y + \overline{uv_y} - \bar{u}\,\bar{v}_y\,. \end{aligned}$$

Reversing all the signs, we have

$$F_1^{\mathrm{R}} = -\overline{[(u-\bar{u})^2]_x} - \overline{[(u-\bar{u})(v-\bar{v})]_y}\,, \tag{2.9}$$

with a symmetric derivation giving

$$F_2^{\mathrm{R}} = -\overline{[(v-\bar{v})^2]_y} - \overline{[(u-\bar{u})(v-\bar{v})]_x}\,. \tag{2.10}$$

It follows from (2.9), (2.10) that $\mathbf{F}^{\mathrm{R}}$ has zero 2D spatial mean, while $\mathbf{F}^\partial$ might not. We decompose the averaged velocity as $\mathbf{u}_2 = \bar{\mathbf{u}} - \bar{\mathbf{u}}_0$, where

$$\bar{\mathbf{u}}_0 = \frac{1}{L^2}\int_{\overline{\Omega}} \bar{\mathbf{u}}\,d\bar{\mathbf{x}}, \qquad \bar{\mathbf{x}} = (x,y)$$