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Visibility graphs and symbolic dynamics

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H I G H L I G H T S

- We explore the relation between visibility graphs and symbolic dynamics.
- Graph theoretic entropies converge to the Lyapunov exponent.
- We construct the induced effective phase space partition.

A R T I C L E I N F O

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A B S T R A C T

Visibility algorithms are a family of geometric and ordering criteria by which a real-valued time series of N data is mapped into a graph of N nodes. This graph has been shown to often inherit in its topology nontrivial properties of the series structure, and can thus be seen as a combinatorial representation of a dynamical system. Here we explore in some detail the relation between visibility graphs and symbolic dynamics. To do that, we consider the degree sequence of *horizontal visibility graphs* generated by the one-parameter logistic map, for a range of values of the parameter for which the map shows chaotic behaviour. Numerically, we observe that in the chaotic region the block entropies of these sequences systematically converge to the Lyapunov exponent of the time series. Hence, Pesin's identity suggests that these block entropies are converging to the Kolmogorov–Sinai entropy of the physical measure, which ultimately suggests that the algorithm is implicitly and adaptively constructing phase space partitions which might have the generating property. To give analytical insight, we explore the relation $k(x)$, $x \in [0, 1]$ that, for a given datum with value x, assigns in graph space a node with degree k. In the case of the *out*-degree sequence, such relation is indeed a piece-wise constant function. By making use of explicit methods and tools from symbolic dynamics we are able to analytically show that the algorithm indeed performs an effective partition of the phase space and that such partition is naturally expressed as a countable union of subintervals, where the endpoints of each subinterval are related to the fixed point structure of the iterates of the map and the subinterval enumeration is associated with particular ordering structures that we called motifs.

1. Introduction

The family of visibility algorithms [1,2] are a set of simple criteria by which ordered real-valued sequences – and in particular, time series – can be mapped into graphs, thereby allowing the inspection of dynamical processes using the tools of graph theory. In recent years research on this topic has essentially focused on two different fronts: from a theoretical perspective, some works have focused on providing a foundation to these transformations [3–5], while in other cases authors have explored the resulting combinatorial analogues of some well-known dynamical measures [6,7]. Similarly, the graph-theoretical description of canonical routes to chaos [8–11] and some classical stochastic processes [6,12] have been discussed recently under this approach, as well as the exploration of relevant statistical properties such as time irreversibility [13,14]. From an applied perspective, these methods are routinely used to describe in combinatorial and topological terms experimental signals emerging in different fields including physics [15–21], neuroscience [22–25] or finance [26] to cite a few examples where analysis and classification of such signals is relevant. In addition it is worth to notice that the methodology is highlighted as well in JRC technical reports [27,28] which provide evidence-based scientific support to the policy making process of the European Commission.

Here we consider in some detail the so-called *horizontal visibility graph* (HVG) associated to paradigmatic examples of nonlinear, chaotic dynamics and we focus on a specific property of the graphs, namely the degree sequence, a set that lists the degree of each node. As will be shown below, HVGs inherit the time arrow of the

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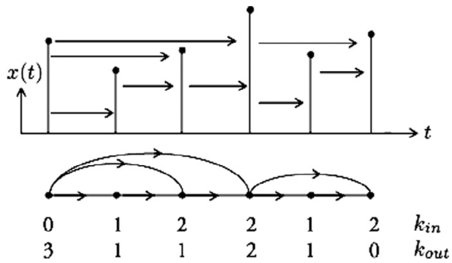

Fig. 1. Illustration of the process of constructing a horizontal visibility graph (HVG) from a time series. If one associates a time arrow to the links, we can decompose the degree of each node into an in-degree k_{in} and an *out*-degree k_{out}, making the HVG a directed graph.

associated time series and therefore their degree sequences are naturally ordered according to this time arrow. Since the degree of a node is an integer quantity, the degree series $\{k_t\}_{t=1}^{t_{max}}$ of an HVG can be seen as an integer representation of the associated time series $\{x_t\}_{t=1}^{t_{max}}$, i.e. as a symbolised series. However, such symbolisation is far from obvious, as a priori there is no explicit partition of the state space which provides such a symbolisation. In this work we explore this problem from the perspective of dynamical systems theory, and more particularly we explore the connections between HVGs and symbolic dynamics. After briefly presenting the simple horizontal visibility algorithm in Section 2, in Section 3 we explore the statistical properties of the degree sequence when constructed from a chaotic logistic map $x_{t+1} = rx_t(1 - x_t)$. We give numerical evidence that the block entropies over the degree sequence converge to the Lyapunov exponent of the map for all the values of the parameter r for which the map shows chaotic behaviour.

In formal terms a dynamical system comes with an invariant measure. Since we adopt a viewpoint from time series analysis we consider what is frequently called the physical measure or the SRB measure of the dynamical system. Quantities such as the Lyapunov exponent or the Kolmogorov–Sinai entropy are hence meant with respect to such a measure. In such an ergodic theoretic context HVGs can be considered as random graphs with the seed x_0 being distributed according to the associated invariant measure. In the case of SRB measures the Kolmogorov–Sinai entropy (a metric dynamical invariant of the map) finds a combinatorial analogue defined in terms of the statistics of the degree sequence. This matching further suggests that the degree sequence is indeed produced after an effective symbolisation of the system's trajectories. In Section 4 we explore by analytical means a possible effective partition of the phase space which could produce such symbolisation. Finally, in Section 5 we close the article with some discussion.

2. Horizontal visibility graphs

The visibility algorithms [1,2] are a family of rules to map a real-valued time series $\{x_t\}_{t=1}^{t_{\max}}$, $x_t \in \mathbb{R}$ into graphs (the multivariate version has been proposed recently [29]). In the *horizontal visibility* case [2], each datum x_t in the time series is associated with a vertex t in the horizontal visibility graph (HVG) $\mathcal{G}$ inducing a natural vertex ordering in the graph. Two vertices i and j are connected by an edge in $\mathcal{G}$ if (see Fig. 1)

$$x_k < \inf(x_i, x_j)\,\forall k:\; i < k < j. \tag{1}$$

Geometrically, two vertices share an edge if the associated data are larger than any intermediate data. Given a node i, we say that node j is *visible* to i if there exists a link between i and j, otherwise j is a *hidden* node wrt i. $\mathcal{G}$ can be characterised as a noncrossing outerplanar graph with a Hamiltonian path [4]. Statistically, $\mathcal{G}$ is an order statistic [12] as it does not depend on the series marginal distribution.

Recent results suggest that the topological properties of $\mathcal{G}$ capture nontrivial structures of the time series, in such a way that graph theory can be used to describe classes of different dynamical systems in a succinct and novel way, including graph-theoretical descriptions of canonical routes to chaos [9–11], discrimination between stochastic and chaotic dynamics [7]. Interestingly, note that the degree of node i can be further split into $k_i = k_i^{\text{in}} + k_i^{\text{out}}$, where the *in*-degree k_i^{in} accounts for all the edges that node i shares with nodes j for which $j < i$ (so called past nodes), and where the *out*-degree k_i^{out} accounts for all the edges that node i shares with nodes j for which $j > i$ (so called future nodes). While node i can a priori inherit a link from any past node, in practice the oldest node which can send a link to a node i is the closest one whose associated data height is larger than or equal to x_i. Similarly, node i's out-degree is bounded by the first node $j > i$ whose associated height supersedes x_i. Hence in- and out-degrees of a site are well defined quantities which only depend on a finite neighbourhood of the site. For instance, the penultimate node in Fig. 1 has in- and out-degree one, no matter how the rest of the time series looks like, as the node is invisible to any node further to the left and does not see any node further to the right. The splitting into in- and out-degree and its relation to the time arrow of the series allows to study the presence of time irreversibility in both deterministic and stochastic dynamical systems [12,13].

Here, we pay particular attention to the degree sequences of $\mathcal{G}$ in the context of low dimensional chaotic dynamics, where $\{x_t\}_{t=1}^{t_{\max}}$, $x_t \in [a, b]$. As commented before, the degree sequence of a graph is a set containing the degree k of each node (total number of incident edges to a given node), $\{k_i\}_{i=1}^{t_{\max}}$, where $k_i \in \mathbb{N}$, so it is a purely topological property of $\mathcal{G}$. Empirical evidence suggests that this is a very informative feature and it was recently proved [5] that, under mild conditions, there exists a bijection between the degree sequence and the adjacency matrix of an HVG. In other words the degree sequence often encapsulates all the complexity of the graph, so this sequence is a good candidate to account for the associated series complexity.

In HVGs, the time arrow indeed induces a natural ordering on the degree sequence where k_i is the degree of node i and nodes are ordered according to the natural time order. Thus, in some sense one may see $\{k_i\}$ as a coarse-grained symbolic representation of the time series. However, it is far from clear if $\{k_i\}$ results from any *effective* partitioning of the state space $[a, b]$ into a set of non-overlapping subsets: the algorithm itself does not partition the phase space explicitly. Furthermore, the number of different symbols k_i is not determined a priori, as depending on the particular dynamics underlying the time series under study, the number of different degrees, i.e., the number of symbols might vary arbitrarily. Even worse, there does not seem to exist a unique transformation between the series datum x_i and its associated node degree k_i: each x_i may have a different associated symbol depending on the position of x_i in the series. In this sense it is not straightforward at all to identify $\{k_i\}$ as a symbolic dynamics of the map. We shall explore these matters in detail, and we will show that we can indeed link a (non-standard) symbolic dynamics with the *out*-degree sequence $\{k_i^{\text{out}}\}_{i=1}^{t_{\max}}$. Before we can explore these aspects in detail we want to recall some background tools in symbolic dynamics, for the convenience of the reader. In addition, we will provide as well some numerical evidence.

3. Entropies

3.1. Symbolic dynamics and chaotic one dimensional maps

The concept of entropy, originally introduced in thermodynamics, has become one of the most prominent measures to quantify