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Geometric chaos indicators and computations of the spherical hypertube manifolds of the spatial circular restricted three-body problem

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HIGHLIGHTS

- Global computation of the center manifolds in the spatial circular restricted three-body problem.
- Formulation of variational theory related to Kustaanheimo–Stiefel regularization.
- Definition of suited geometric chaos indicators.
- Application to the Sun–Jupiter system.

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ABSTRACT

The circular restricted three-body problem has five relative equilibria $L_1, L_2, \ldots, L_5$. The invariant stable–unstable manifolds of the center manifolds originating at the partially hyperbolic equilibria L_1, L_2 have been identified as the separatrices for the motions which transit between the regions of the phase-space which are internal or external with respect to the two massive bodies. While the stable and unstable manifolds of the planar problem have been extensively studied both theoretically and numerically, the spatial case has not been as deeply investigated. This paper is devoted to the global computation of these manifolds in the spatial case with a suitable finite time chaos indicator. The definition of the chaos indicator is not trivial, since the mandatory use of the regularizing Kustaanheimo–Stiefel variables may introduce discontinuities in the finite time chaos indicators. From the study of such discontinuities, we define geometric chaos indicators which are globally defined and smooth, and whose ridges sharply approximate the stable and unstable manifolds of the center manifolds of L_1, L_2. We illustrate the method for the Sun–Jupiter mass ratio, and represent the topology of the asymptotic manifolds using sections and three-dimensional representations.

1. Introduction

The circular restricted three-body problem describes the motion of a massless body P in the gravitation field of two massive bodies P_1 and P_2, called primary and secondary body respectively, which rotate uniformly around their common center of mass. In a rotating frame the problem has five equilibria, the so called Lagrangian points $L_1, \ldots, L_5$, which are the only known simple solutions of the equations of motion of P:

$$\begin{cases} \ddot{x} = 2\dot{y} + x - (1-\mu)\dfrac{x+\mu}{r_1^3} - \mu\dfrac{x-1+\mu}{r_2^3} \\ \ddot{y} = -2\dot{x} + y - (1-\mu)\dfrac{y}{r_1^3} - \mu\dfrac{y}{r_2^3} \\ \ddot{z} = -(1-\mu)\dfrac{z}{r_1^3} - \mu\dfrac{z}{r_2^3}, \end{cases} \quad (1)$$

where the units of masses, lengths and time have been chosen so that the masses of P_1 and P_2 are $1-\mu$ and μ ($\mu \le 1/2$) respectively, their coordinates are $(-\mu, 0, 0)$ and $(1-\mu, 0, 0)$ and their revolution period is 2π; we denoted by $r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}$ and by $r_2 = \sqrt{(x-1+\mu)^2 + y^2 + z^2}$ the distances of P from P_1, P_2. As it

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is well known, Eqs. (1) have an integral of motion, the so—called Jacobi constant, defined by:

$$\mathcal{C}(x, y, z, \dot{x}, \dot{y}, \dot{z}) = x^2 + y^2 + 2\frac{1-\mu}{r_1} + 2\frac{\mu}{r_2} - \dot{x}^2 - \dot{y}^2 - \dot{z}^2. \quad (2)$$

Fixed values C of the Jacobi constant define level sets $\mathcal{M}_C$ in the phase-space, which project on the set:

$$\Pi\mathcal{M}_C = \{(x, y, z) \in \mathbb{R}^3_0 : x^2 + y^2 + 2\frac{1-\mu}{r_1} + 2\frac{\mu}{r_2} \geq C\}$$

of the physical space:

$$\mathbb{R}^3_0 = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (-\mu, 0, 0), (1-\mu, 0, 0)\}.$$

The boundary $\mathcal{B}_C$ of $\Pi\mathcal{M}_C$ separates the so called realm of possible motions $\Pi\mathcal{M}_C$ from the realm of forbidden motions $\mathbb{R}^3_0 \backslash \Pi\mathcal{M}_C$.

The Lagrangian equilibria $L_1, \ldots, L_5$ are critical points for the Jacobi constant; the values $C_1, C_2, \ldots, C_5$ of $\mathcal{C}$ at the Lagrangian equilibria $L_1, \ldots, L_5$ correspond to topological changes of the set $\mathcal{B}_C$. In particular, for $C > C_2$ the space $\mathbb{R}^3_0$ is disconnected by $\mathcal{B}_C$ into a region of motions which contains the massive bodied P_1, P_2 and an external region; for $C < C_2$ the realm of possible motions is connected; in particular, for values of C slightly smaller than C_2, the connection between the internal and external regions is realized through a bottleneck of $\mathcal{B}_C$, at whose extremities we find the Lagrangian points L_1 and L_2. The transit of motions through this bottleneck is guided by the stable–unstable manifolds of the center manifolds W^c_1, W^c_2 originating at the equilibria L_1, L_2, which are partially hyperbolic, specifically they are saddle×center×center. The center manifold theorem (see, for example, [1]) grants the existence of two four-dimensional center manifolds W^c_i, $i = 1, 2$. Since the restriction of the Jacobi constant to each W^C_i has a strict extremum at the equilibrium point L_i, from the general results of [1] we obtain that for suitably small values of $C - C_i$ the sets $W^c_{C,i} = W^c_i \cap \mathcal{M}_C$ are unique, are diffeomorphic to a three-sphere, are invariant with respect to the flow of the three-body problem for any time $t \in \mathbb{R}$. Their stable and unstable manifolds have the topology of hypertubes obtained from the product of a three-sphere with a half line; we will call them spherical hypertube manifolds. The spherical hypertube manifolds act as separatrices for the transit of motions through the bottlenecks of $\mathcal{B}_C$ connecting the region of internal and the region of external motions, see [2–4] (for the planar three-body problem) and [5,6] (for the spatial three-body problem). This fact is a consequence of the structure of the local stable–unstable manifolds of $W^c_{C,i}$ in a small neighborhood $\mathcal{U}_i$: motions with initial conditions in $\mathcal{U}_i$ approaching the center manifold from the right-hand side (left-hand side respectively) 'bounce back' if they are on one side of the separatrix, while they transit to the left-hand side (right-hand side respectively) if they are on the other side of the separatrix.

The structure of the global stable and unstable manifolds of $W^c_{C,i}$ is much more complicate than the structure of the local manifolds: the exponential compressions, expansions and rotations occurring near the center manifolds are alternated to circulations around both primaries. Global representations of these surfaces have been obtained for several sample values of μ and C in the planar circular restricted three-body problem, see for example [4,7,8]. The computation of the stable–unstable manifolds in the planar case has several advantages with respect to the spatial case. First, in the planar case, the level set of the center manifolds obtained by fixing the value of the Jacobi constant in a suitable small left neighborhood of C_i is made of a periodic orbit, the horizontal Lyapunov orbit of L_1 or L_2. To compute their asymptotic manifolds one can use one of the several methods of computation of the stable and unstable manifolds of periodic orbits, for example the flow continuation of the local manifolds, the parametrization method, or the recent method based on chaos indicators (see [7] and the references therein). Moreover, the phase-space of the planar three-body problem is four dimensional, and by fixing the value of the Jacobi constant we obtain a three dimensional space. The stable and unstable manifolds of the horizontal Lyapunov orbits are therefore two-dimensional surfaces in a three-dimensional space; their global phase-space development has been graphically visualized in [8]. For the spatial case, in the literature there are computations of stable and unstable manifolds of orbits in the center manifolds, like the halo and Lissajous orbits, obtained from high order semi-analytical expansions (see [9]), or using set oriented procedures (see [10]).

Different methods to compute stable and unstable manifolds of invariant objects use chaos indicators. The finite time chaos indicators, such as the finite time Lyapunov exponent (FTLE hereafter) and the fast Lyapunov indicator (FLI hereafter), originally defined in [11,12], have been used in the last decade to compute numerical approximations of the stable and unstable manifolds of equilibria, periodic orbits, and the so called Lagrangian coherent structures of turbulent flows of many model systems (see for example [13–24]). In [7] we have shown that the traditional finite time chaos indicators can fail completely the detection of the stable or unstable manifolds of hyperbolic equilibria or periodic orbits. The problem has been solved with a major modification to the chaos indicator, by taking into account for its computation only the contributions from the variational equations due to a neighborhood of the hyperbolic fixed point or periodic orbit; i.e. by filtering out all the other contributions. The extension of the method introduced in [7,8] to the spatial restricted three-body problem requires additional major modifications for three reasons: (i) the center manifold $W^s_{C,i}$ has dimension 3 in a 5-dimensional reduced phase-space (while in the planar problem is just a periodic orbit in a 3 dimensional reduced phase-space) and does not contain only periodic or quasi-periodic orbits; (ii) the regularization of the spatial problem is geometrically more complicate than the planar Levi-Civita regularization; (iii) the efficient computation of the filtered chaos indicator is intrinsically more complicate for a center manifold than for a periodic orbit. In this paper we solve these problems by considering a family of chaos indicators, modified as follows. First, we construct a neighborhood $\mathcal{U}_i$ of the center manifold $W^c_{C,i}$ where the local stable and unstable manifolds are represented as Cartesian graphs and a hyperbolic Birkhoff normal form of some convenient order is defined; then, we localize the global stable manifold $W^s_{C,i}$ by exploiting at the same time the peculiar growth of the tangent vectors close to $W^c_{C,i}$, and the scattering from $\mathcal{U}_i$ of the motions with initial conditions outside the local stable manifold. Both properties are coded by a filtered finite time chaos indicator, fast Lyapunov indicator or finite time Lyapunov indicator, whose ridges provide the stable manifold[1] $W^s_{C,i}$.

The definition of smooth chaos indicators for the spatial three-body problem is non trivial since Eqs. (1) are singular for $(x, y, z) = (-\mu, 0, 0)$ or $(1-\mu, 0, 0)$. Smooth indicators will be defined by using the Kustaanheimo–Stiefel regularization which has been originally introduced to regularize the spatial problem (KS hereafter, [25,26]). The use of regularizing transformations, with their analytic and computational advantages (see, for example, [27,28]) appears to us mandatory to compute long pieces of the spherical hypertube manifolds of L_1, L_2, especially close to the secondary body. In this spirit, the Levi-Civita transformation (LC hereafter), which regularizes the equations of motion of the planar three-body problem [29], has been used in [7,8] to define chaos indicators whose ridges approximate the stable and unstable manifolds of the horizontal Lyapunov orbits of L_1, L_2. Even if the KS transformation is the natural extension of the LC transformation, the regularization

[1] As usual, the unstable manifold $W^u_{C,i}$ is obtained by computing the stable manifold of the inverse flow.