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Two-dimensional models for the combined bending and stretching of plates and shells based on three-dimensional linear elasticity

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Abstract

Models for plates and shells derived from three-dimensional linear elasticity, based on a thickness-wise expansion of the strain energy of a thin body, are described. These involve the small thickness explicitly and accommodate combined bending and stretching in a single framework. Physically motivated local constraints on the through-thickness variation of the displacement field, required for consistency with the exact theory, are introduced. When incorporated into the energy functional, these yield an expression for the two-dimensional strain energy density that includes non-standard two-dimensional strain gradient effects.

Keywords: Plates; Shells; Linear elasticity; Strain gradients

> Extreme rigour in the analysis of physical problems, we are inclined to believe, may easily lead to rigor mortis. ...Flexible bodies like thin shells require a flexible approach. ...In a difficult physical problem like shell theory there is both room and a need for intuition and imagination in addition to mathematical analysis.
> Koiter [1].

1. Introduction

The derivation of an accurate two-dimensional model of plates and shells from three-dimensional elasticity theory has been a problem of recurring interest over the course of the history of solid mechanics [2–11]. In recent years, the problem has been addressed through the use of sophisticated tools such as the method of Gamma convergence, aimed at generating the two-dimensional variational problem in the limit of small

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thickness, or methods of asymptotic analysis, in which leading-order (in thickness) weak forms of the equations are obtained via a process of formal expansion. A discussion of the current state of the subject is summarized concisely in the recent work of Ciarlet [12], which contains an extensive bibliography. To date, these methods have yielded sharp results for pure membrane behavior and for inextensional bending, but not for the practically important intermediate case in which bending and stretching occur simultaneously. In particular, neither method has generated a single model containing the small thickness explicitly, in contrast to the situation for classical theories. Such a structure is to be expected in any model that accommodates bending and stretching simultaneously. The limitations of models derived from Gamma convergence or asymptotic expansions of the weak forms of the equilibrium equations are inherent in these methods. For example, Gamma convergence entails passage to the limit of zero thickness, and so generates only the leading-order model appropriate for the considered loads and boundary conditions. The asymptotic approach is similar in that it requires successive terms in the expansion to be negligibly small in comparison to the preceding terms. Accordingly, as remarked in [13], these methods, despite their limited success, effectively separate the scales associated with membrane and bending behavior to a greater extent than desired. An important exception in this regard is the model of Koiter [14,15], which Ciarlet [12] advocates as the best practical solution to the problem of specifying the equations holding in the interior of the shell in the case of small strains with possibly finite displacements. This despite the fact that Koiter's theory is not a limit model in the sense of Gamma convergence or asymptotic expansions.

The present paper is concerned with the derivation of models for combined bending and stretching in the presence of edge conditions. The approach followed is based on a systematic small thickness expansion of the exact three-dimensional strain energy density of the plate or shell in which the thickness figures explicitly. This may be truncated at any desired level. The equilibrium equations for the truncated model are the Euler equations of the associated energy integral. Membrane effects are associated with the order h problem, where h is the thickness, and corrections, associated with bending and additional non-standard effects, emerge at order h^3 if the midsurface is a plane of symmetry of the material properties. Our model does not suffer from the overly restrictive separation of scales that characterizes the limit models discussed above. Nevertheless, in any particular problem it is in principle necessary to use its predictions to evaluate the leading-order terms not retained to ensure that those appearing explicitly are dominant.

Also considered is a locally constrained variant of the model motivated by comparison with the equations of three-dimensional elasticity. The resulting strain energy density for the midsurface contains non-standard two-dimensional strain gradient terms in addition to the usual stretching and bending energies. These are suppressed in classical plate theory. This finding is not unexpected. Indeed, it is known that such terms are present in the thickness-wise expansions of the equilibrium energies of a thin plate in plane stress or generalized plane stress [16, Arts. 310, 303, 304]. The extra terms are associated with a singular perturbation. Accordingly, their effect may be expected to be confined to a thin layer adjoining an edge, necessary to accommodate assigned or reactive force and moment distributions.

The development of plate theory is described at length in Section 2. In Section 3 we adapt the ideas developed for plates to shells. While the basic concepts are preserved, some reconsideration is necessary to accommodate the differential geometry of the shell. To hold the discussion to a tractable level, and to make contact with a large class of applications, we base our development on the classical linear theory of elasticity. However, our methods easily generalize to non-linear elasticity.

Standard notation is used throughout. Thus, bold face is used for vectors and tensors and indices are used to denote their components. Latin indices take values in {1, 2, 3}; Greek in {1, 2}. The latter are associated with surface coordinates and associated vector and tensor components. A dot between bold symbols is used to denote the standard inner product. Thus, if $\mathbf{A}_1$ and $\mathbf{A}_2$ are second-order tensors, then $\mathbf{A}_1 \cdot \mathbf{A}_2 = \operatorname{tr}(\mathbf{A}_1\mathbf{A}_2^t)$, where $\operatorname{tr}(\cdot)$ is the trace and the superscript $(^t)$ is used to denote the transpose. The norm of a tensor $\mathbf{A}$ is $|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}}$. The linear operator $\operatorname{Sym}(\cdot)$ delivers the symmetric part of its second-order tensor argument. The notation $\otimes$ identifies the standard tensor product of vectors. If $\mathbf{C}$ is a fourth-order tensor, then $\mathbf{C}[\mathbf{A}]$ is the second-order tensor with orthogonal components $C_{ijkl}A_{kl}$. Finally, we use symbols such as Div and D to denote the three-dimensional divergence and gradient operators, while div and ∇ are reserved for their two-dimensional counterparts. Thus, for example, $\operatorname{Div}\mathbf{A} = A_{ij,j}\mathbf{e}_i$ and $\operatorname{div}\mathbf{A} = A_{i\alpha,\alpha}\mathbf{e}_i$, where $\{\mathbf{e}_i\}$ is an orthonormal basis and subscripts preceded by commas are used to denote partial derivatives with respect to Carte-