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Pseudo-simple heteroclinic cycles in $\mathbb{R}^4$

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HIGHLIGHTS

- Finite subgroups of $O(4)$ admitting pseudo-simple heteroclinic cycles are identified.
- Existence of periodic orbits close to pseudo-simple cycles is investigated.
- Subgroups of $O(4) \setminus SO(4)$ admitting f.a.s. pseudo-simple cycles are identified.

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ABSTRACT

We study *pseudo-simple* heteroclinic cycles for a Γ-equivariant system in $\mathbb{R}^4$ with finite $\Gamma \subset O(4)$, and their nearby dynamics. In particular, in a first step towards a full classification – analogous to that which exists already for the class of *simple* cycles – we identify all finite subgroups of $O(4)$ admitting pseudo-simple cycles. To this end we introduce a constructive method to build equivariant dynamical systems possessing a robust heteroclinic cycle. Extending a previous study we also investigate the existence of periodic orbits close to a pseudo-simple cycle, which depends on the symmetry groups of equilibria in the cycle. Moreover, we identify subgroups $\Gamma \subset O(4)$, $\Gamma \not\subset SO(4)$, admitting fragmentarily asymptotically stable pseudo-simple heteroclinic cycles. (It has been previously shown that for $\Gamma \subset SO(4)$ pseudo-simple cycles generically are completely unstable.) Finally, we study a generalized heteroclinic cycle, which involves a pseudo-simple cycle as a subset.

1. Introduction

A heteroclinic cycle is an invariant set of a dynamical system comprised of equilibria $\xi_1, \ldots, \xi_M$ and heteroclinic orbits κ_i from ξ_i to ξ_{i+1}, $i = 1 \ldots M$ with the convention $M + 1 = 1$. For several decades these objects have been of keen interest to the nonlinear science community. A heteroclinic cycle is associated with intermittent dynamics, where the system alternates between states of almost stationary behaviour and phases of quick change. It is well-known that a heteroclinic cycle can exist robustly in equivariant dynamical systems, i.e. persist under generic equivariant perturbations, namely when all heteroclinic orbits are saddle–sink connections in (flow-invariant) fixed-point subspaces. Robust heteroclinic cycles, their nearby dynamics and attraction properties have been thoroughly studied, especially in low dimensions. See [1,2] for a general overview. In $\mathbb{R}^3$, there are comparatively few possibilities for heteroclinic dynamics and these are rather well-understood. In $\mathbb{R}^4$, the situation is significantly more involved. We therefore consider systems

$$\dot{x} = f(x), \tag{1}$$

where $f : \mathbb{R}^4 \to \mathbb{R}^4$ is a smooth map that is equivariant with respect to the action of a finite group $\Gamma \subset O(4)$, i.e.

$$f(\gamma x) = \gamma f(x) \quad \text{for all } x \in \mathbb{R}^4,\ \gamma \in \Gamma. \tag{2}$$

In this setting, much attention has been paid to so-called *simple* cycles, see e.g. [3–6], for which (i) all connections lie in two-dimensional fixed-point spaces $P_j = \mathrm{Fix}(\Sigma_j)$ with $\Sigma_j \subset \Gamma$, and (ii) the cycle intersects each connected component of $P_{j-1} \cap P_j \setminus \{0\}$ at most once. This definition was introduced by [6], who also suggested several examples of subgroups of $O(4)$ that admit such a cycle (in the sense that there is an open set of Γ-equivariant vector fields possessing such an invariant set). The classification of simple cycles was completed in [7,8] (for homoclinic cycles) and finally in [9] by finding all groups $\Gamma \subset O(4)$ admitting such a cycle. In [9] it was also discovered that the original definition of simple cycles from [6] implicitly assumed a condition on the isotypic decomposition of $\mathbb{R}^4$ with respect to the isotropy subgroup

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of an equilibrium, see Section 2.1 for details. This prompted them to define *pseudo-simple* heteroclinic cycles as those satisfying (i) and (ii) above, but not this implicit condition.

It is the primary aim of the present paper to carry out a systematic study of pseudo-simple cycles in $\mathbb{R}^4$, by establishing a complete list of all groups $\Gamma \subset O(4)$ that admit such a cycle. This is done in a similar fashion to the classification of simple cycles in [9], by using a quaternionic approach to describe finite subgroups of $O(4)$. First examples for pseudo-simple cycles were investigated in [9,10]. The latter of those also addressed stability issues: it was shown that a pseudo-simple cycle with $\Gamma \subset SO(4)$ is generically completely unstable, while for the case $\Gamma \not\subset SO(4)$ a cycle displaying a weak form of stability, called *fragmentary asymptotic stability*, was found. A fragmentarily asymptotically stable (f.a.s.) cycle has a positive measure basin of attraction that does not necessarily include a full neighbourhood of the cycle [11]. We extend this stability study by showing an example of group $\Gamma \not\subset SO(4)$ which admits an asymptotically stable generalized heteroclinic cycle and pseudo-simple subcycles that are f.a.s. Moreover, we look at the dynamics near a pseudo-simple cycle and discover that asymptotically stable periodic orbits may bifurcate from it. Whether or not this happens depends on the isotropy subgroup $\mathbb{D}_k$, $k \geq 3$ of equilibria comprising the cycle. The case $k = 3$ was already considered in [10]. We illustrate our more general results by numerical simulations for an example with $\Gamma = (\mathbb{D}_4 \,|\, \mathbb{D}_2; \mathbb{D}_4 \,|\, \mathbb{D}_2)$ in the case $k = 4$.

This paper is organized as follows. Section 2 recalls background information on (pseudo-simple) heteroclinic cycles and useful properties of quaternions as a means to describe finite subgroups of $O(4)$. Then, in Section 3 we give conditions that allow us to decide whether or not such a group $\Gamma \subset O(4)$ admits pseudo-simple heteroclinic cycles. Section 4 contains the statement and proofs of Theorems 1 and 2, which use the previous results to list all subgroups of $O(4)$ admitting pseudo-simple heteroclinic cycles. The proof of Theorem 1 relies on properties of finite subgroups of $SO(4)$ that are given in the Appendix. In Section 5 we investigate the existence of asymptotically stable periodic orbits close to a pseudo-simple cycle, depending on the symmetry groups $\mathbb{D}_k$, of equilibria. The cases $k = 3, 4$ and $k \geq 5$ are covered by Theorems 3 and 4, respectively. In Section 6 we employ the ideas of the previous sections to provide a numerical example of a pseudo-simple cycle with a nearby attracting periodic orbit. Finally, in Section 7 for a family of subgroups $\Gamma \not\subset SO(4)$ we construct a generalized heteroclinic cycle (i.e., a cycle with multidimensional connection(s)) and prove conditions for its asymptotic stability in Theorem 5. This cycle involves as a subset a pseudo-simple heteroclinic cycle, that can be fragmentarily asymptotically stable. Section 8 concludes and identifies possible continuations of this study. The appendix contains additional information on subgroups of $SO(4)$ that is relevant for the proof of Theorem 1.

2. Background

Here we briefly review basic concepts and terminology for pseudo-simple heteroclinic cycles and the quaternionic approach to describing subgroups of $SO(4)$ as needed in this paper.

2.1. Pseudo-simple heteroclinic cycles

In this subsection we give the precise framework in which we investigate robust heteroclinic cycles and the associated dynamics. Given an equivariant system (1) with finite $\Gamma \subset O(4)$ recall that for $x \in \mathbb{R}^4$ the *isotropy subgroup of* x is the subgroup of all elements in Γ that fix x. On the other hand, given a subgroup $\Sigma \subset \Gamma$ we denote by $\mathrm{Fix}\,(\Sigma)$ its *fixed point space*, i.e. the space of points in $\mathbb{R}^4$ that are fixed by all elements of Σ.

Let $\xi_1, \ldots, \xi_M$ be hyperbolic equilibria of a system (1) with stable and unstable manifolds $W^s(\xi_j)$ and $W^u(\xi_j)$, respectively. Also, let $\kappa_j \subset W^u(\xi_j) \cap W^s(\xi_{j+1}) \neq \varnothing$ for $j = 1, \ldots, M$ be connections between them, where we set $\xi_{M+1} = \xi_1$. Then the union of equilibria $\{\xi_1, \ldots, \xi_M\}$ and connecting trajectories $\{\kappa_1, \ldots, \kappa_M\}$ is called a *heteroclinic cycle*. Following [5] we say it is *structurally stable* or *robust* if for all j there are subgroups $\Sigma_j \subset \Gamma$ such that ξ_{j+1} is a sink in $P_j := \mathrm{Fix}\,(\Sigma_j)$ and κ_j is contained in P_j. We also employ the established notation $L_j := P_{j-1} \cap P_j = \mathrm{Fix}\,(\Delta_j)$, with a subgroup $\Delta_j \subset \Gamma$. As usual we divide the eigenvalues of the Jacobian $df(\xi_j)$ into *radial* (eigenspace belonging to L_j), *contracting* (belonging to $P_{j-1} \ominus L_j$), *expanding* (belonging to $P_j \ominus L_j$) and *transverse* (all others), where we write $X \ominus Y$ for a complementary subspace of Y in X. In accordance with [5] our interest lies in cycles where

- (H1) $\dim P_j = 2$ for all j,
- (H2) the heteroclinic cycle intersects each connected component of $L_j \setminus \{0\}$ at most once.

Then there is one eigenvalue of each type and we denote the corresponding contracting, expanding and transverse eigenspaces of $df(\xi_j)$ by V_j, W_j and T_j, respectively. In [9] it is shown that under these conditions there are three possibilities for the unique Δ_j-isotypic decomposition of $\mathbb{R}^4$:

1. $\mathbb{R}^4 = L_j \oplus V_j \oplus W_j \oplus T_j$
2. $\mathbb{R}^4 = L_j \oplus V_j \oplus \widetilde{W}_j$, where $\widetilde{W}_j = W_j \oplus T_j$ is two-dimensional
3. $\mathbb{R}^4 = L_j \oplus W_j \oplus \widetilde{V}_j$, where $\widetilde{V}_j = V_j \oplus T_j$ is two-dimensional.

Here $\oplus$ denotes the orthogonal direct sum. This prompts the following definition.

Definition 1 (*[9]*). We call a heteroclinic cycle satisfying conditions (H1) and (H2) above *simple* if case 1 holds true for all j, and *pseudo-simple* otherwise.

Remark 1. In case 1 the group Δ_j acts as $\mathbb{Z}_2$ on each one-dimensional component other than L_j and $\Delta_j \cong \mathbb{D}_2$ (which is always the case if $\Gamma \subset SO(4)$) or $\Delta_j \cong (\mathbb{Z}_2)^3$. In cases 2 and 3 the group acts on the two-dimensional isotypic component as a dihedral group $\mathbb{D}_k$ in $\mathbb{R}^2$ for some $k \geq 3$ and $\Delta_j \cong \mathbb{D}_k = \langle \rho_j, \sigma_j \rangle$ (always for $\Gamma \subset SO(4)$) or $\Delta_j \cong \mathbb{D}_k \times \mathbb{Z}_2$. For $\Gamma \subset SO(4)$ in case 2 the element ρ_j acts as a k-fold rotation on $\widetilde{W}_j$ and trivially on $P_{j-1} = L_j \oplus V_j$, while σ_j acts as $-I$ on $V_j \oplus T_j$ and trivially on $L_j \oplus W_j$. In case 3 the element ρ_j acts as a k-fold rotation on $\widetilde{V}_j$ and trivially on $P_j = L_j \oplus W_j$, while σ_j acts as $-I$ on $W_j \oplus T_j$ and trivially on $L_j \oplus V_j$.

Remark 2. The existence of a two-dimensional isotypic component implies that in case 2 the contracting and transverse eigenvalues are equal ($c_j = t_j$) and the associated eigenspace is two-dimensional, while in case 3 the expanding and transverse eigenvalues are equal ($e_j = t_j$) and the associated eigenspace is two-dimensional. Hence, we say that $df(\xi_j)$ has a multiple contracting or expanding eigenvalue in cases 2 or 3, respectively.

We are interested in identifying all subgroups of $O(4)$ that admit pseudo-simple heteroclinic cycles in the following sense. For simple cycles this task has been achieved step by step in [6–9].

Definition 2 (*[9]*). We say that a subgroup Γ of $\mathrm{O}(n)$ *admits* (pseudo-)simple heteroclinic cycles if there exists an open subset of the set of smooth Γ-equivariant vector fields in $\mathbb{R}^n$, such that all vector fields in this subset possess a (pseudo-)simple heteroclinic cycle.

In order to establish the existence of a heteroclinic cycle it is sufficient to find a sequence of connections $\xi_1 \to \cdots \to \xi_{m+1} = \gamma\xi_1$ with some finite order $\gamma \in \Gamma$, that is minimal in the sense that no $i, j \in \{1, \ldots, m\}$ satisfy $\xi_i = \gamma'\xi_j$ for any $\gamma' \in \Gamma$.