ARTICLE IN PRESS

Physica D ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Scale-free behavior of networks with the copresence of preferential and uniform attachment rules

Angelica Pachon *, Laura Sacerdote, Shuyi Yang

Mathematics Department "G. Peano", University of Torino, Italy

HIGHLIGHTS

- A variant of the preferential attachment model which takes into account two different attachment rules is proposed.
- These rules are: a preferential attachment mechanism and a uniform choice only for the most recent nodes.
- The recent nodes can be either a given fixed number or a proportion of the total number of existing nodes.
- We prove that this model exhibits an asymptotically power-law degree distribution.

ARTICLE INFO

Article history:
Received 28 April 2017
Received in revised form 8 January 2018
Accepted 10 January 2018
Available online xxxx
Communicated by S. Coombes

Keywords:
Barabási–Albert random graph
Preferential and uniform attachments
Degree distribution

ABSTRACT

Complex networks in different areas exhibit degree distributions with a heavy upper tail. A preferential attachment mechanism in a growth process produces a graph with this feature. We herein investigate a variant of the simple preferential attachment model, whose modifications are interesting for two main reasons: to analyze more realistic models and to study the robustness of the scale-free behavior of the degree distribution.

We introduce and study a model which takes into account two different attachment rules: a preferential attachment mechanism (with probability $1-p$) that stresses the rich get richer system, and a uniform choice (with probability p) for the most recent nodes, i.e. the nodes belonging to a window of size w to the left of the last born node. The latter highlights a trend to select one of the last added nodes when no information is available. The recent nodes can be either a given fixed number or a proportion (αn) of the total number of existing nodes. In the first case, we prove that this model exhibits an asymptotically power-law degree distribution. The same result is then illustrated through simulations in the second case. When the window of recent nodes has a constant size, we herein prove that the presence of the uniform rule delays the starting time from which the asymptotic regime starts to hold.

The mean number of nodes of degree k and the asymptotic degree distribution are also determined analytically. Finally, a sensitivity analysis on the parameters of the model is performed.

1. Introduction

Networks grow according to different paradigms and the preferential attachment mechanism is one of the simplest rules that explains some of the observed features exhibited by real networks. Barabási–Albert proposed it to model the World-Wide Web (WWW) [1]. However, it is also used for a variety of other phenomena such as citation networks or some genetic networks [2–4]. The preferential attachment model connects new and existing nodes with probabilities proportional to the number of links already present. This rule is also known as rich get richer because it rewards with new links nodes with a higher number of incoming links. Networks growing according with the preferential attachment rule exhibit the *scale free property* and their degree distribution is characterized by power law tails [1,5,6].

Other models for different real world networks request the use of different growth paradigms and do not present the scale free property. For example, some networks exhibit the small-world phenomenon, in which sub-networks have connections between almost any two nodes within them. Furthermore, most pairs of nodes are connected by at least one short path, see [7] for a comprehensive account of complex networks from a physics perspective. On the other hand, one of the most studied models, the Erdös–Rényi random graph, does not exhibit either the power-law behavior for the degree distribution of its nodes, nor the small-world phenomenon [8–12].

* Corresponding author.
E-mail address: apachonp@unito.it (A. Pachon).

https://doi.org/10.1016/j.physd.2018.01.005

Mathematically, the growth of networks can be modeled through random graph processes, i.e. a family $(G^t)_{t\in\mathbb{N}}$ of random graphs (defined on a common probability space), where t is interpreted as time. Different features of the model are then described as properties of the corresponding random graph process. In particular, the interest often focuses on the *degree*, $\deg(v, t)$, of a vertex v at time t, that is on the total number of incoming and (or) outgoing edges to and (or) from v, respectively. In this framework, new nodes of the Barabási–Albert model link with higher probabilities with nodes of higher degree.

An important feature of preferential attachment models is an asymptotic power-law degree distribution: the fraction $P(k)$ of vertices in the network with degree k, goes as $P(k) \sim k^{-\gamma}$, with $\gamma > 0$, for large values of k.

Real world modeling instances motivated the proposal of generalizations of the Barabási–Albert model, see e.g. [13–22]. Other models are for example, the "altruistic" attachment rule introduced in [23] or [24] where the authors consider networks in which the increase of the connectivity of the nodes depends on their fitness to compete for links. A common characteristic of many of these models is the presence of the same attachment rule for all the nodes of the network. However, this hypothesis is not always realistic. Consider, for example, websites where registered members can submit contents and vote for them, such as text and posts. Typically, the participants tend to vote either the most popular or the most recent posts (see www.reddit.com). Hence, the user votes the posts according to two different rules: with uniform probability if the user decides to select a post recently published, and with probability proportional to the number of votes, otherwise. A similar structure of network growth arises also when we consider some social networks. A new subject may connect choosing uniformly from people who recently joined the group (for instance new schoolmates) or preferring famous people present in the network since a long period of time (for example schoolmates belonging to a rock band). A third instance arises in the case of citations. Typically, authors of a new paper cite recent work on the same subject as well as the most important papers on the considered topic.

Having this type of networks in our mind, in Section 2 we propose a generalization of the Barabási–Albert model which takes into account the two different attachment rules for new nodes of the network. We called it *Uniform-Preferential-Attachment* model (UPA model) to pinpoint the double nature of the attachment rule. In [25], Magner et al. introduced a model in which new vertices choose the nodes for their links within windows. Inspired by their idea of introducing windows, we formulate our model. We apply the preferential attachment rule to any node but we re-inforce this rule with a uniform choice for the most recent nodes. To define recent nodes we introduce windows either of fixed or linear in time amplitude. Our first result in Section 3 is the proof of recursive formulas for the expectation of the number of nodes with a given degree at a fixed time, in the case of windows of fixed amplitude. Our study takes advantage of the existing methods used for example in [6,26], together with the Azuma–Hoeffding Inequality (see Lemma 4.2.3 in [11]), but some of the relationships requested by these techniques are not trivial in our case. Since the mixing of uniform and preferential attachment rules suggests the possible disappearance of the scale free property we investigated this possibility. In Section 3, we also determine the degree distribution by employing a rigorous mathematical approach in the case of fixed size windows. The use of this distribution allowed us to prove the asymptotic scale free property. These results are illustrated in Section 4 through a sensitivity analysis for the different parameters. Furthermore, in Section 4 we use simulations to study the case in which the size of the windows grows linearly in time. We show that an asymptotically power-law degree distribution is preserved.

Finally, Section 5 contains some concluding remarks, while the Appendix reports some auxiliary lemmas and their proofs.

2. Uniform-Preferential-Attachment model (UPA model) and related models

2.1. The UPA model

In order to model instances in which recent nodes play also an important role, we propose that every new node v_{t+1} selects its neighbor within a limited window $\{v_{t-w+1}, \dots, v_t\}$ of nodes of size $w \in \mathbb{N}$ (i.e. the w youngest nodes of the network) or among all nodes $\{v_0, \dots, v_t\}$. The former happens with probability $p \in [0, 1]$ and the latter with probability $(1 - p)$. Furthermore, the attachment rules are different in these two cases. When there is the window, the neighbor is chosen with a uniform distribution (that is, every node within the window has probability $\frac{1}{w}$ to be selected), but in absence of window, the neighbor is chosen according to a preferential attachment mechanism (that is $v_j, j = 0, \dots, t$ has a probability proportional to $\deg(v_j, t)$ to be chosen).

We first fix a probability $0 \le p \le 1$ and a natural number $l \ge 1$. The process starts at time $t = l$, with $l + 1$ adjacent vertices, growing monotonically by adding at each discrete time step a new vertex v_{t+1} together with a directed edge connecting this with some of the vertices already present. As far as the window size is concerned we consider two cases:

1. For all $t \ge l$, $w := w(t) = l$, $l \in \mathbb{N}$, that is the window size is fixed; and
2. for each $t \ge l$, $w := w(t) = \lceil \alpha t \rceil$, $0 < \alpha < 1$, that is the size of the window is a linear function of the size of the network.

We study the first case analytically. Instead, for the second case we limit ourselves to numerical results. In the following we refer to this second case as the *generalization* of the UPA model. The algorithmic description of the UPA model is:

(a) At the starting period $t = l$, $l \in \mathbb{N}$, the initial graph G^l has $l + 1$ nodes $(v_0, v_1, \dots v_l)$, where every node v_j, $1 \le j \le l$, is connected to v_0.
(b) Given G^t, at time $t + 1$ add a new node v_{t+1} together with an outgoing edge. Such edge links v_{t+1} with an existing node chosen either within a window, or among all nodes present in the network at time t, as follows:
   - with probability p, v_{t+1} chooses its neighbor in the set $\{v_{t-w+1}, \dots, v_t\}$, and each node within this window has probability $\frac{1}{w}$ of being chosen.
   - with probability $1 - p$, the neighbor of v_{t+1} is chosen from the set $\{v_0, \dots, v_t\}$, and each node v_j, $j = 0, \dots, t$, has probability $\frac{\deg(v_j)}{2t}$ of being chosen.

   Here, $\deg(v_j)$ indicates the total number of incoming links to v_j.

Remark 2.1. Note that:

- When $p = 0$ the UPA model reduces to the usual preferential attachment model of Barabási–Albert [1].
- The initial degrees of the nodes of the UPA model, are $\deg(v_j) = 1$ for $1 \le j \le l$ and $\deg(v_0) = l$.

2.2. Related models

The UPA model recalls a very general model suggested by Cooper and Frieze [15] involving preferential and uniform attachment rules. In their model, at each step, a new vertex appears with probability $1 - \alpha$ and generates i edges with probability p_i. On the other hand, with probability α an old vertex is chosen, such that with probability δ it is chosen uniformly between the

Please cite this article in press as: A. Pachon, et al., Scale-free behavior of networks with the copresence of preferential and uniform attachment rules, Physica D (2018), https://doi.org/10.1016/j.physd.2018.01.005.