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Concentration and limit behaviors of stationary measures

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HIGHLIGHTS

- Noise vanishing limits of stationary measures of Fokker–Planck equations are studied.
- Invariance of the limit measures to the ODE system is shown.
- Local concentrations of stationary measures in the vicinity of a local attractor or repeller are characterized.
- Noise stabilization of a local attractor and de-stabilization of a local repeller are shown with respect to particular noises.
- Noise de-stabilization of a repelling equilibrium is shown with respect to general noises.

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ABSTRACT

In this paper, we study limit behaviors of stationary measures of the Fokker–Planck equations associated with a system of ordinary differential equations perturbed by a class of multiplicative noise including additive white noise case. As the noises are vanishing, various results on the invariance and concentration of the limit measures are obtained. In particular, we show that if the noise perturbed systems admit a uniform Lyapunov function, then the stationary measures form a relatively sequentially compact set whose weak*-limits are invariant measures of the unperturbed system concentrated on its global attractor. In the case that the global attractor contains a strong local attractor, we further show that there exists a family of admissible multiplicative noises with respect to which all limit measures are actually concentrated on the local attractor; and on the contrary, in the presence of a strong local repeller in the global attractor, there exists a family of admissible multiplicative noises with respect to which no limit measure can be concentrated on the local repeller. Moreover, we show that if there is a strongly repelling equilibrium in the global attractor, then limit measures with respect to typical families of multiplicative noises are always concentrated away from the equilibrium. As applications of these results, an example of stochastic Hopf bifurcation and an example with non-decomposable ω-limit sets are provided.

Our study is closely related to the problem of noise stability of compact invariant sets and invariant measures of the unperturbed system.

1. Introduction

Regarded as a physical model, a dynamical system generated from ordinary differential equations is often subject to noise perturbations either from its surrounding environment or from intrinsic uncertainties associated with the system. Analyzing the impact of noise perturbations on the dynamics of the system then becomes a fundamental issue with respect to both modeling and dynamics.

There have been many studies toward this dynamics issue using either a trajectory-based or a distribution-based approach. The trajectory-based approach is often adopted under the framework of random dynamical systems, i.e., skew-product flows with ergodic measure-preserving base flows. By assuming vanishing noise at a reference equilibrium, noise perturbations of essential dynamics of a dynamical system are studied under the random dynamical system framework with respect to problems such as noise perturbations of invariant manifolds [1–4], normal forms

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[5–8], and stochastic bifurcations (see [5] and references therein). For a system of ordinary differential equations subject to white noise perturbations vanishing at a reference equilibrium, we refer the reader to [9] for some study of stochastic stability of the equilibrium (see also [10] for similar studies in infinite dimension).

With respect to general noise perturbations, the distribution-based approach is useful and seemly necessary to adopt under both frameworks of random dynamical systems and Itô stochastic differential equations. Due to its essential differences from deterministic dynamical systems, much less is known in this direction comparing with cases using the trajectory-based approach. For some import pioneer works on noise perturbations of dynamical systems on a compact manifold from the viewpoint of distributions, we refer the reader to [11,12] for stochastic stability of flows on a 2-torus or a periodic cycle, to [13] for stochastic stability of essentially a finite number of equilibria and periodic cycles by introducing large deviation theory, to [14–17] for stochastic stability of SRB measures, and to [18] for some global stochastic stability characterizations.

In this paper, we adopt the distribution-based approach to study the impact of white noises on basic dynamics of a system of ordinary differential equations in an Euclidean space. More precisely, we consider a system of ordinary differential equations

$$\dot{x} = V(x), \qquad x \in \mathcal{U} \subset \mathbb{R}^n, \tag{1.1}$$

where $\mathcal{U}$ is a connected open set which can be bounded, unbounded, or the entire $\mathbb{R}^n$, and $V = (V^i) \in C(\mathcal{U}, \mathbb{R}^n)$. We assume throughout the paper that (1.1) generates a local flow φ^t on $\mathcal{U}$. The generality of domain $\mathcal{U}$ does allow a wide range of applications because many physical models (e.g., those concerning populations and concentrations) are not necessarily defined in the entire $\mathbb{R}^n$. Adding general multiplicative (i.e., spatially non-homogeneous) including additive (i.e., spatially homogeneous) white noise perturbations, we obtain the following Itô stochastic differential equations (SDE in short)

$$\mathrm{d}x = V(x)\mathrm{d}t + G(x)\mathrm{d}W, \qquad x \in \mathcal{U} \subset \mathbb{R}^n, \tag{1.2}$$

where W is a standard m-dimensional Brownian motion for some integer $m \geq n$, and $G = (g^{ij})_{n\times m}$ is a matrix-valued function on $\mathcal{U}$, called *noise coefficient matrix*. For generality, we assume that $g^{ij} \in W^{1,p}_{loc}(\mathcal{U})$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, for some fixed constant $p > n$.

The stochastic differential equations (1.2) arise naturally as a non-isolated physical system subject to noise perturbations from its surrounding environments, in which the impact of noises on dynamics is often physically measured in term of distributions. They can also arise naturally from the study of a large scale deterministic but seemingly stochastic system, for instance a so-called mesoscopic system which is partially structured but contains intrinsic uncertainties in a fast time scale due to high complexity, large degree of freedom, lack of full knowledge of mechanisms, the need for organizing a large amount of data, etc. Under some exponential mixing assumptions on the fast dynamics, such a mesoscopic system can have a stochastic reduction of the form (1.2) over any finite time interval in which V represents the structured field and G encompasses all dynamical uncertainties (see e.g., [19,20]). Very recently, such stochastic reduction theory is systematically extended to the case of partially hyperbolic fast–slow systems (see e.g., [21–23]). It has been argued for a mesoscopic system that in the case of sufficiently high uncertainty, a trajectory-based approach using either deterministic or random dynamics modeling would not provide much information to its dynamical description. Instead, a distribution-based approach using stochastic differential equations like (1.2) is necessary to adopt in order to synthesize the typical patterns of dynamics (see [24] and references therein).

An important distribution-based approach for studying diffusion process generated by (1.2) is to use its associated *Fokker–Planck equation* (also called *Kolmogorov forward equation*)

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = L_A u(x,t), \quad x \in \mathcal{U},\ t > 0, \\ u(x,t) \geq 0, \quad \displaystyle\int_{\mathcal{U}} u(x,t)\mathrm{d}x = 1, \end{cases} \tag{1.3}$$

where $A = (a^{ij}) = \frac{GG^\top}{2}$, called the *diffusion matrix*, and L_A is the Fokker–Planck operator defined as

$$L_A g(x) = \partial^2_{ij}(a^{ij}(x)g(x)) - \partial_i(V^i(x)g(x)), \qquad g \in C^2(\mathcal{U}).$$

We note that $a^{ij} \in W^{1,p}_{loc}(\mathcal{U})$, $i, j = 1, 2, \ldots, n$. It is well-known that if the stochastic differential equation (1.2) generates a (local) diffusion process in $\mathcal{U}$ (e.g., when both V and G are locally Lipschitz in $\mathcal{U}$), then its transition probability density function, if it exists, is actually a (local) fundamental solution of the Fokker–Planck equation (1.3).

In the above and also through the rest of the paper, we use short notions $\partial_i = \frac{\partial}{\partial x_i}$, $\partial^2_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$, and we also adopt the usual summation convention on $i, j = 1, 2, \ldots, n$ whenever applicable.

Long time behaviors of solutions of the Fokker–Planck equation (1.3) is governed by the stationary Fokker–Planck equation

$$\begin{cases} L_A u = \partial^2_{ij}(a^{ij}u) - \partial_i(V^i u) = 0, \\ u(x) \geq 0, \quad \displaystyle\int_{\mathcal{U}} u(x)\mathrm{d}x = 1, \end{cases} \tag{1.4}$$

which, in the weak form, becomes

$$\begin{cases} \displaystyle\int_{\mathcal{U}} \mathcal{L}_A f(x) u(x)\mathrm{d}x = 0, \qquad \text{for all } f \in C_0^\infty(\mathcal{U}), \\ u(x) \geq 0, \quad \displaystyle\int_{\mathcal{U}} u(x)\mathrm{d}x = 1, \end{cases} \tag{1.5}$$

where $C_0^\infty(\mathcal{U})$ denotes the space of C^∞ functions on $\mathcal{U}$ with compact supports and

$$\mathcal{L}_A = a^{ij}\partial^2_{ij} + V^i\partial_i$$

is the adjoint Fokker–Planck operator corresponding to A. Solutions of (1.5) are called *weak stationary solutions of* (1.3) or *stationary solutions corresponding to* $\mathcal{L}_A$. More generally, one considers a measure-valued stationary solution μ_A of the Fokker–Planck equation (1.3), called a *stationary measure of the Fokker–Planck equation* (1.3) or a *stationary measure corresponding to* $\mathcal{L}_A$, which is a Borel probability measure satisfying

$$\int_{\mathcal{U}} \mathcal{L}_A f(x)\mathrm{d}\mu_A(x) = 0, \qquad \text{for all } f \in C_0^\infty(\mathcal{U}). \tag{1.6}$$

If a stationary measure μ_A is regular, i.e., $\mathrm{d}\mu_A(x) = u_A(x)\mathrm{d}x$ for some density function $u_A \in C(\mathcal{U})$, then it is clear that u_A is necessarily a weak stationary solution of (1.3), i.e, it satisfies (1.5). Conversely, according to the regularity theorem in [25], if (a^{ij}) is everywhere positive definite in $\mathcal{U}$, then any stationary measure corresponding to $\mathcal{L}_A$ must be regular with positive density function lying in $W^{1,p}_{loc}(\mathcal{U})$. In the case that (1.2) generates a diffusion process on $\mathcal{U}$, it is well-known that any invariant measure of the diffusion process is necessarily a stationary measure of the Fokker–Planck equation (1.3), but the converse need not be true. However, under some mild conditions a stationary measure of the Fokker–Planck equation (1.3) is always a sub-invariant measure of some generalized diffusion process (see [26–28] for discussions in this regard in particular with respect to the uniqueness of stationary measures and their invariance). In this sense, stationary measures of (1.3) may be regarded as generalizations of invariant measures of a classical diffusion process.