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TURING PATTERNS IN PARABOLIC SYSTEMS OF CONSERVATION LAWS AND NUMERICALLY OBSERVED STABILITY OF PERIODIC WAVES

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Abstract. Turing patterns on unbounded domains have been widely studied in systems of reaction-diffusion equations. However, up to now, they have not been studied for systems of conservation laws. Here, we (i) derive conditions for Turing instability in conservation laws and (ii) use these conditions to find families of periodic solutions bifurcating from uniform states, numerically continuing these families into the large-amplitude regime. For the examples studied, numerical stability analysis suggests that stable periodic waves can emerge either from supercritical Turing bifurcations or, via secondary bifurcation as amplitude is increased, from subcritical Turing bifurcations. This answers in the affirmative a question of Oh-Zumbrun whether stable periodic solutions of conservation laws can occur. Determination of a full small-amplitude stability diagram– specifically, determination of rigorous Eckhaus-type stability conditions– remains an interesting open problem.

1. Introduction

The study of periodic solutions of conservation laws and their stability, initiated in [OZ03a, OZ03b] and continued in [Ser05, JZ10], etc., has led to a number of interesting developments, particularly in the related study of roll-waves in inclined shallow-water flow. For an account of these developments, see, e.g., [JNRZ12] and references therein. However, in the original context of conservation laws, so far *no example of a stable periodic wave has been found.* Indeed, one of the primary results of [OZ03a, PSZ13] was that for the fundamental example of planar viscoelasticity, stable periodic waves do not exist, due to a special variational structure of this particular system; it was cited as a basic open problem whether stable periodic waves could arise for *any* system of conservation laws, either physically motivated: or artificially contrived.

In the more standard context of reaction diffusion systems and classical pattern formation theory, by contrast, stable periodic solutions are abundant and well-understood, through the mechanism of *Turing instability*, or bifurcation of small-amplitude, approximately-constant period, periodic solutions from a uniform state. For such waves, stability is completely determined by an associated *Eckhaus stability diagram*, as derived formally in [Eck65] and verified rigorously in [Mie95, Mie97, Sch96, SZJV16], essentially by perturbation from constant-coefficient linearized behavior. By contrast, the small-amplitude waves investigated up to now (see Remark 3.2) come through more complicated zero-wave number bifurcations in which period goes to infinity as amplitude goes to zero and the stability analysis is far from constant-coefficient (see, e.g., [Bar14] in the successfully-analyzed case of shallow-water flow).

Our simple goal in this paper, therefore, is *to seek stable periodic waves via a conservation law analog of Turing instability.* In the first part, we find an analog of Turing instability, with which we are able to generate large numbers of examples of spatially periodic solutions of conservation laws. Next, we find an interesting dimensional restriction to systems of three or more coordinates,

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