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Travelling waves and their bifurcations in the Lorenz-96 model

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HIGHLIGHTS

- The existence of Hopf and Hopf-Hopf bifurcations is proven for all $n \geq 4$.
- An analytical formula is derived for the first Lyapunov coefficient of the Hopf bifurcation.
- Periodic attractors have the physical interpretation of travelling waves.
- The Hopf-Hopf bifurcation acts as an organising center and leads to multistability.
- The dynamics beyond the first bifurcation depends on n, but without clear pattern.

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ABSTRACT

In this paper we study the dynamics of the monoscale Lorenz-96 model using both analytical and numerical means. The bifurcations for positive forcing parameter F are investigated. The main analytical result is the existence of Hopf or Hopf–Hopf bifurcations in any dimension $n \geq 4$. Exploiting the circulant structure of the Jacobian matrix enables us to reduce the first Lyapunov coefficient to an explicit formula from which it can be determined when the Hopf bifurcation is sub- or supercritical. The first Hopf bifurcation for $F > 0$ is always supercritical and the periodic orbit born at this bifurcation has the physical interpretation of a travelling wave. Furthermore, by unfolding the codimension two Hopf–Hopf bifurcation it is shown to act as an organising centre, explaining dynamics such as quasi-periodic attractors and multistability, which are observed in the original Lorenz-96 model. Finally, the region of parameter values beyond the first Hopf bifurcation value is investigated numerically and routes to chaos are described using bifurcation diagrams and Lyapunov exponents. The observed routes to chaos are various but without clear pattern as $n \to \infty$.

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1. Introduction

1.1. Setting of the problem

In his 1996 paper [1], Edward Lorenz introduced two models to study fundamental issues regarding the predictability of the atmosphere and weather forecasting. The so-called monoscale Lorenz-96 model is defined by the equations

$$\dot{x}_j = x_{j-1}(x_{j+1} - x_{j-2}) - x_j + F, \qquad j = 1, \ldots, n, \tag{1a}$$

where we take the indices modulo n by the following 'boundary condition'

$$x_{j-n} = x_{j+n} = x_j, \tag{1b}$$

resulting in a model with circulant symmetry. For the multiscale model, which will not be discussed in this paper, the reader is referred to [1]. The model (1) can be interpreted as a model for atmospheric waves travelling along a circle of constant latitude. Lorenz interpreted the variables x_j as values of some meteorological quantity (e.g., temperature, pressure, or vorticity) in n equal sectors of a latitude circle, where the index j plays the role of longitude. The continuous parameter F represents external forcing and can be used as a bifurcation parameter.

Although the Lorenz-96 model is not derived from physical principles it still has features which are commonly found in geophysical models: forcing, dissipation and energy preserving quadratic terms. Moreover, unlike the traditional Lorenz-63 model [2] which has only one positive Lyapunov exponent, the Lorenz-96 model has multiple positive Lyapunov exponents for suitable choices of the parameters F and n. For those reasons, and for the simplicity of the equations, the model is important and widely used nowadays and sometimes even called "the archetype of large deterministic systems displaying chaotic behaviour" [3] or

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Table 1
Recent papers with applications of the monoscale Lorenz-96 model (1) and the main values of n and F that were used. Almost all values are chosen in the chaotic domain ($F = 8$) of dimension $n = 36$ or 40.

Reference	Application	n	F
Danforth & Yorke [12]	Making forecasts	40	8
Dieci et al. [13]	Approximating Lyapunov exponents	40	8
Gallavotti & Lucarini [14]	Non-equilibrium ensembles	32	≥8
Hansen & Smith [15]	Operational constraints	40	8
De Leeuw et al. [16]	Data assimilation	36	8
Lorenz & Emanuel [17]	Optimal sites	40	8
Lorenz [18]	Designing chaotic models	30	10, 15 (2.5, …, 40)
Lorenz [1]	Predictability	36 (4)	8 (15, 18)
Lucarini & Sarno [19]	Ruelle linear response theory	40	8
Ott et al. [20]	Data assimilation	40, 80, 120	8
Stappers & Barkmeijer [21]	Adjoint modelling	40	8
Sterk et al. [22]	Predictability of extremes	36	8
Sterk & Van Kekem [23]	Predictability of extremes	4, 7, 24	11.85, 4.4, 3.85
Trevisan & Palatella [24]	Data assimilation	40, 60, 80	8

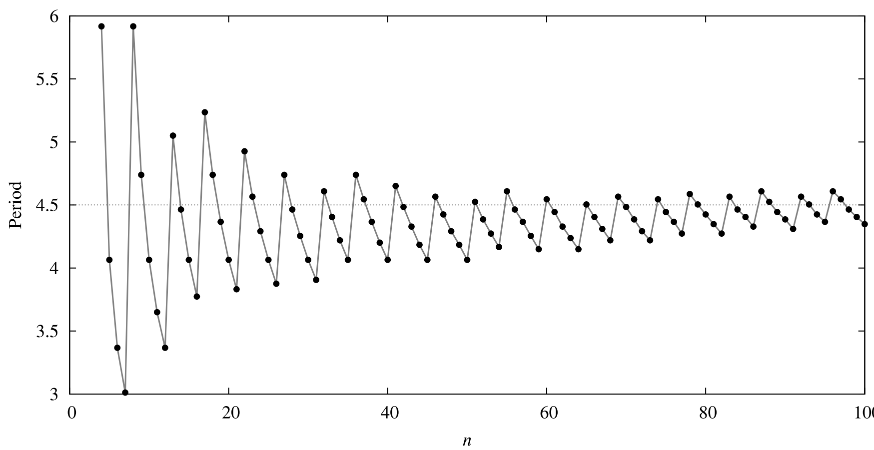

Fig. 1. The period of the periodic attractor of the Lorenz-96 model detected for $F = 1.2$ plotted as a function of the dimension n. Note that the period converges to approximately 4.5 as $n \to \infty$. See Fig. 4 for a comparison with the theoretical period at the Hopf bifurcation.

"a hallmark representative of nonlinear dynamical systems" [4]. The applications of the Lorenz-96 model are broad and range from geophysical applications like data assimilation and predictability to studies in spatiotemporal chaos. Table 1 gives an overview of recent papers in which the Lorenz-96 has been used together with the values of the parameters that were used.

In contrast to its importance, only a few studies have investigated the dynamics of this model. In [5], the high-dimensional chaotic dynamics has been explored by means of the fractal dimension. A recent study on patterns of order and chaos in the multiscale model has reported the existence of regions with standing waves [4]. Bifurcation diagrams in low dimensions of the Lorenz-96 model have been studied in [6], although the emphasis of their work was on methods to visualise bifurcations by means of spectral analysis, rather than exploring the dynamics itself. The previous works already revealed an extraordinarily rich structure of the dynamical behaviour of the Lorenz-96 model for specific values of n. However, there has been no systematic study of the dynamics of this model yet. In this paper we fill this gap by studying the dynamical nature of the Lorenz-96 model in greater detail and give analytical proofs of some basic properties for all dimensions and of the existence of Hopf and Hopf–Hopf bifurcations. These results are complemented by numerical explorations, that includes the dynamics beyond these bifurcations as well.

The Lorenz-96 model is a *family* of dynamical systems parameterised by the discrete parameter $n \in \mathbb{N}$ which gives the dimension of their state space. This setup is analogous to a discretised partial differential equation. In fact, in some works the Lorenz-96 model is interpreted as such [7,8]. In [9], a discretised quasi-geostrophic model for the atmosphere was studied. In particular, they numerically observed that the parameter value at which the first Hopf bifurcation occurs typically increases with the truncation order of their discretisation method. In pseudo-spectral discretisations of Burgers' equation [10] qualitative differences in dynamics were observed depending on whether the dimension of state space was even or odd. We may expect similar phenomena for the Lorenz-96 model. Hence, in this paper we focus in particular on the question which quantitative and qualitative features of the dynamics will persist for (almost) all $n \in \mathbb{N}$. Answers to these questions may be helpful in selecting appropriate values of n and F for the specific applications listed in Table 1. For example, there is a direct relation between the dimension of attractors and the statistics of extreme events [11]. Although the study of this paper is unable to provide the entire picture, it offers a partial dynamical inventory using both analytical and numerical means.

1.2. Sketch of the results

The Lorenz-96 model (1) has an equilibrium solution given by $x_F = (F, \dots, F)$ for all $n \geq 1$ and $F \in \mathbb{R}$. Clearly, for $F = 0$ this equilibrium is stable. Numerical simulations show that for $F = 1.2$ the dynamics of the model is periodic for all $n \geq 4$. This suggests that for $0 < F < 1.2$ a supercritical Hopf bifurcation occurs at which the equilibrium x_F loses its stability and gives birth to a periodic attractor. Fig. 1 shows that the period of the periodic attractor at $F = 1.2$ is an oscillating function of the dimension n. Observe that the oscillations decay with n and that the period seems to converge to a value of approximately 4.5. The spatiotemporal properties of

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