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On the manifestation of coexisting nontrivial equilibria leading to potential well escapes in an inhomogeneous floating body

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HIGHLIGHTS

- Stability analysis is presented for an inhomogeneous floating body.
- Bifurcation diagrams and basins of attraction reveal complex stability behavior.
- Static experiments are conducted to verify numerical results.
- Dynamic experiments are conducted to investigate potential well hopping.

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ABSTRACT

This paper examines the bifurcation and stability behavior of inhomogeneous floating bodies, specifically a rectangular prism with asymmetric mass distribution. A nonlinear model is developed to determine the stability of the upright and tilted equilibrium positions as a function of the vertical position of the center of mass within the prism. These equilibria positions are defined by an angle of rotation and a vertical position where rotational motion is restricted to a two dimensional plane. Numerical investigations are conducted using path-following continuation methods to determine equilibria solutions and evaluate stability. Bifurcation diagrams and basins of attraction that illustrate the stability of the equilibrium positions as a function of the vertical position of the center of mass within the prism are generated. These results reveal complex stability behavior with many coexisting solutions. Static experiments are conducted to validate equilibria orientations against numerical predictions with results showing good agreement. Dynamic experiments that examine potential well hopping behavior in a waveflume for various wave conditions are also conducted.

1. Introduction

A classic engineering rule of thumb states that a floating object can be made stable in the upright position if its center of mass is located below the centroid of the displaced fluid, or the center of buoyancy. This fact can be proven by showing that the center of buoyancy shifts to the right for clockwise rotation about the center of mass and to the left for counter-clockwise rotation which results in a restoring torque for both cases. More sophisticated analysis has been conducted in the field of Naval Architecture to show that this upright position can remain stable even if the center of gravity is located above the center of buoyancy by introducing the concept of a metacenter [1]. The metacenter is defined as the point of intersection between a vertical line drawn through the center of buoyancy when the body is tilted and a vertical line drawn through the original center of buoyancy at equilibrium. It has been shown that a floating object will remain stable even if the center of mass is located above the center of buoyancy as long as the metacenter is located above the center of mass [2]. The analysis to prove this linearizes the system about its equilibrium point to determine whether small perturbations will result in an instability commonly called capsizing. This criterion has been well documented and is extremely useful for Naval Engineers in designing hull shapes, prescribing maximum payloads, and constructing ballasts [3].

While this work can be applied to inform design decisions, scientists and engineers have examined the stability of floating bodies on a more fundamental level in attempt to gain deeper phenomenological insights. These studies have looked to investigate the equilibrium and stability behavior of various symmetric floating objects and have been particularly interested in the emergence of non-trivial tilted equilibrium positions [4–6]. Gilbert investigated the equilibria of homogeneous cylinders and tetrahedrons for various specific density and geometric ratios to illustrate the emergence of tilted equilibrium positions [7]. Rorres expanded on

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this work by conducting similar studies on the more complicated paraboloid [8]. Erdos et al. then progressed the field even further by showing how the same non-trivial equilibria states could be realized for inhomogeneous shapes by simply shifting the location of the object's center of mass [9].

These studies offer useful insights into the behavior of floating bodies, however little investigation has been done to examine scenarios in which multiple non-trivial equilibrium solutions coexist simultaneously. Intuition suggests that tilted equilibrium solutions should be possible for scenarios in which the center of mass is located off the prism's centerline, however, it is not intuitive whether this is possible for a symmetric mass distribution and, if so, which of these solutions would be stable. Furthermore, if situations can be identified where coexisting stable equilibrium positions exist, would it be possible to transition back and forth between stable orientations through dynamic wave excitation? If such conditions could be determined, this behavior could be manipulated for a host of hydrodynamic engineering applications [10]. This paper seeks to answer these questions by examining the bifurcation characteristics of a floating rectangular prism (hereafter referred to as a buoy) to identify and study coexisting tilted equilibrium positions in floating bodies. In particular, this paper develops a nonlinear mathematical model to determine the static stability of the upright and tilted equilibria positions as a function of the vertical position of the center of mass within the buoy, validates the numerical results through experimental studies, and then applies this knowledge to examine the buoy's behavior under dynamic wave excitation.

The remainder of this paper is organized as follows. Section 2 defines the geometry and fundamental variables used to describe the problem. In doing so, six distinct regions that correspond to scenarios where different combinations of corners are submerged below the waterline are identified [6]. Section 3 uses this geometry to derive the governing equations of motion and explain the key parameters involved. Section 4 then describes how these equations are used to determine equilibrium positions and stability through numerical methods. Results are presented for four particular cases with qualitatively different bifurcation behaviors. Finally, Section 5 describes experimental tests that successfully validate numerical equilibrium results for the static case and examine responses where the buoy oscillations hop back and forth between potential wells under dynamic wave conditions.

2. Important geometry

Fig. 1 shows a schematic of an upright and tilted rectangular prism along with its important geometry. As illustrated, five fundamental geometric parameters are used to describe the prism: its height a, width b, length ℓ, mass m, and vertical distance from the base to its center of mass k. Analyzing what happens when k is varied for a prism with a fixed set of parameters a, b, ℓ, and m is this paper's primary focus. From these parameters, the submerged depth f, can be calculated as

$$f = \frac{m}{\ell b \rho}, \tag{1}$$

where ρ represents the density of the surrounding fluid. Beyond these geometric parameters, additional variables are required to describe the position of the buoy as it undergoes planar motion. The buoy's angle of rotation from the upright, vertical displacement, and horizontal displacement are denoted by ϕ, z, and y while corresponding equilibrium positions are given as $\tilde{\phi}$, $\tilde{z}$, and $\tilde{y}$, respectively. These values are described using an inertial reference frame, $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$, with corresponding coordinates x, y, and z, and a body fixed reference frame $(\hat{\boldsymbol{x}}', \hat{\boldsymbol{y}}', \hat{\boldsymbol{z}}')$, with corresponding coordinates x', y', and z', where the origin for both coordinate systems is initially located at the buoy's center of mass.

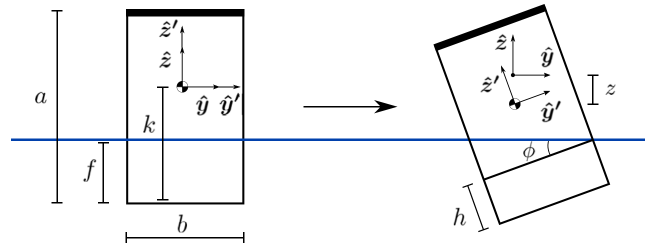

Fig. 1. Geometry in Region 1 (see Fig. 2 for Regions) where $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$ and $(\hat{\boldsymbol{x}}', \hat{\boldsymbol{y}}', \hat{\boldsymbol{z}}')$ denote the inertial and body fixed reference frames, respectively.

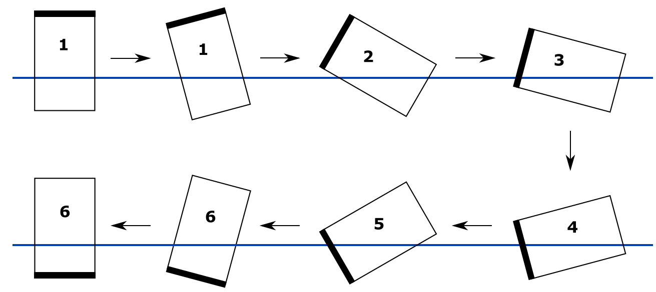

Fig. 2. Schematic of the floating buoy shown in different orientations which shows that a different set of continuous equations is required when different corners are submerged.

As shown in Fig. 2, the motion of the buoy can be separated into six distinct regions depending on the combination of corners that are submerged. As the buoy rotates and translates, new corners become submerged and the differing geometry alters the governing equations of motion [6]. The result is that the buoy's trajectories can be broken into six piecewise smooth regions, each of which are subject to unique equations of motion and physical constraints. The same methods are used to produce solutions over all six regions with three equations adapting as a result of the changing geometry. It is also worth noting that Regions 2 and 5 will look different for the case where $\frac{f}{a} > 0.5$ (i.e. three vertices will be submerged instead of one), however the math remains unchanged. For conciseness, this paper only derives the equations for Region 1 although the same geometric approach outlined here can be used to derive the equations for the five remaining Regions. The complete set of governing equations are used later to generate bifurcation diagrams and can be found in the Appendix.

In describing the system's geometry, the submerged portion can be broken into two parts: a right triangular prism formed by the waterline and a line perpendicular to the buoy's opposite side and a rectangular prism of height h. This geometry is shown in Fig. 3. The value of h can be obtained by determining the vertical component of the position vector from the body fixed origin to the point where the waterline and side of the buoy intersect, $z'(\phi, z, \frac{b}{2})$, and adding this value to the distance between the body fixed origin and the base of the buoy k. Hence, h can be represented as

$$h = k + z'\left(\phi, z, \frac{b}{2}\right). \tag{2}$$

Therefore to determine h it is necessary to write an equation to define $z'(\phi, z, y')$, a point on the waterline as a function of ϕ, z, and y' [6]. This is done using the point-slope formula

$$z' - z'_0 = m(y' - y'_0) \tag{3}$$

where the reference point along the waterline and slope can be expressed as

$$z'_0 = r\cos\phi, \quad y'_0 = r\sin\phi, \quad r = f - z - k, \quad m = -\tan\phi. \tag{4}$$