ARTICLE IN PRESS

Physica D ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Phase models and clustering in networks of oscillators with delayed coupling

Sue Ann Campbell [a], Zhen Wang [b,*]

[a] Department of Applied Mathematics and Centre for Theoretical Neuroscience, University of Waterloo, Waterloo ON N2L 3G1 Canada
[b] Department of Applied Mathematics, University of Waterloo, Waterloo ON N2L 3G1 Canada

ARTICLE INFO

Article history:
Received 2 May 2016
Received in revised form 2 February 2017
Accepted 12 September 2017
Available online xxxx
Communicated by S. Coombes

Keywords:
Time delay
Neural network
Oscillators
Clustering solutions
Stability

ABSTRACT

We consider a general model for a network of oscillators with time delayed coupling where the coupling matrix is circulant. We use the theory of weakly coupled oscillators to reduce the system of delay differential equations to a phase model where the time delay enters as a phase shift. We use the phase model to determine model independent existence and stability results for symmetric cluster solutions. Our results extend previous work to systems with time delay and a more general coupling matrix. We show that the presence of the time delay can lead to the coexistence of multiple stable clustering solutions. We apply our analytical results to a network of Morris Lecar neurons and compare these results with numerical continuation and simulation studies.

1. Introduction

Coupled oscillator models have been used to study many biological and physical systems, for example neural networks [1,2], laser arrays [3,4], flashing of fireflies [5], and movement of a slime mold [6]. A basic question explored with such models is whether the elements in the system will phase-lock, i.e., oscillate with some fixed phase difference, and how the physical parameters affect the answer to this question. Clustering is a type of phase locking behavior where the oscillators in a network separate into groups. Each group consists of fully synchronized oscillators, and different groups are phase-locked with nonzero phase difference. Symmetric clustering refers to the situation when all the groups are the same size while non-symmetric clustering means the groups have different sizes.

A phase model represents each oscillator with a single variable. A differential equation for each phase variable indicates how the phase of the oscillator changes in time:

$$\frac{d\theta_i}{dt} = \Omega_i + H_i(\theta_1, \theta_2, \dots, \theta_N).$$

Here Ω_i is the intrinsic frequency of the ith oscillator and the functions H_i described how the coupling between oscillators influences the phases. Phase models have been used to study the behavior of networks of coupled oscillators beginning with the work of [7]. Phase models are sometimes *posed* as models for coupled oscillators [5,7–9]. When the coupling between oscillators is sufficiently weak, however, a phase model representation of a system can be *derived* from a higher dimensional differential equation model, such as one obtained from a physical or biological description of the system [10–13]. The low dimensional phase model can then be used to predict behavior in the original high dimensional physical model. This approach has proved useful in studying synchronization properties of many different neural models [1,14–20]. Phase models can be linked to experimentally derived phase resetting curves [10,13], thus this approach has also been used to make predictions about synchronization properties of experimental preparations [19].

[21,8] were the first to use phase models to study clustering behavior. Using the theory of equivariant differential equations [21] studied a general network of identical oscillators of arbitrary size with symmetric, weak coupling, corresponding to the symmetry groups S_n, Z_n, and D_n. They determined which type of solutions are forced to exist by the symmetry in each case. For the case of S_n symmetry they gave conditions for the stability of several types of solutions, including symmetric cluster solutions, and determined which bifurcations are forced by symmetry to occur. They also studied the existence of heteroclinic cycles and tori for some special cases. By direct analysis of the phase model, [8] studied a network with global homogeneous coupling, (S_n symmetry). He established general criteria for the stability of all possible symmetric cluster solutions as well as some nonsymmetric cluster

* Corresponding author.
E-mail addresses: sacampbell@uwaterloo.ca (S.A. Campbell), z377wang@uwaterloo.ca (Z. Wang).

http://dx.doi.org/10.1016/j.physd.2017.09.004
0167-2789/

Please cite this article in press as: S.A. Campbell, Z. Wang, Phase models and clustering in networks of oscillators with delayed coupling, Physica D (2017), http://dx.doi.org/10.1016/j.physd.2017.09.004.

solutions. Using numerical simulations, [8] further showed that these results give a good prediction of stability for a variety of model networks. More recently, [22] considered the existence and stability of cluster solutions and fixed tori for phase models corresponding to networks with global homogeneous coupling. They also considered the effect of additional absolute-phase product coupling. Using a similar approach as [8] stability results have been obtained for inhibitory neural networks with nearest-neighbor coupling [23]. Phase model analysis has also been extensively used to study phase-locking in pairs of model and experimental neurons [12,19,24]. More recently it has been used to study clustering in larger neural networks [25,26]. A more comprehensive review of the analysis of phase models and their application to the study of synchronization is given in [27].

In many systems there are time delays in the connections between the oscillators due to the time for a signal to propagate from one element to the other. In neural networks this delay is attributed to the conduction of electrical activity along an axon or a dendrite [12,15]. Much work has been devoted to the study of the effect of time delays in neural networks. However, the majority of this work has focussed on systems where the neurons are excitable not oscillatory, (e.g., [28–33]), the networks have only a few neurons (e.g., [9,12,34–36]) or focussed exclusively on synchronization (e.g., [15,32,37–39]). Extensive work has been done on networks of Stuart–Landau oscillators with delayed diffusive coupling (e.g., [40,41]) where the model for the individual oscillators is the normal form for a Hopf bifurcation and thus the system is often amenable to theoretical analysis. Numerical approaches to study the stability of cluster solutions in delayed neural oscillator networks have also been developed [39,42]. We note that there is a vast literature on time delays in artificial neural networks which we do not attempt to cite here.

Initial studies of phase models for systems with delayed coupling considered models where the delay occurs in the argument of the phases [36,37,43–45]. However, it has been shown [12,46,47] that for small enough time delays it is more appropriate to include the time delay as phase shift in the argument of the coupling function. Crook et al. [15] use this type of model to study a continuum of cortical oscillators with spatially decaying coupling and axonal delay. Bressloff and Coombes [14,48] study phase locking in chains and rings of pulse coupled neurons with distributed delays and show that distributed delays result in phase models with a distribution of phase shifts. They consider phase models derived from integrate and fire neurons and the Kuramoto phase model.

In this paper, we investigate the effect of time delayed coupling on the clustering behavior of oscillator networks. The plan for our article are as follows. In the next section we will review how a general network model with delayed coupling may be reduced to a phase model. In Section 3 we use the phase model to determine conditions for existence and stability of symmetric cluster solutions in a network with a circulant coupling matrix, extending some prior results [8,21,23] to systems with time delays and more general coupling. In Section 4 we consider a particular application: a network of Morris–Lecar oscillators. We derive the particular phase model for this system and compare the predictions of the phase model theory to numerical continuation and simulation studies to determine when the weak coupling assumption breaks down. We show that the time delay can induce multistability between different cluster solutions and explore how changing the coupling matrix affects this. In Section 5 we explore the effects of breaking the symmetry of the connection matrix and introducing multiple time delays on our results. In Section 6 we discuss our work.

2. Reduction to a phase model

In this section, we review how to reduce a general model for a network of all-to-all coupled oscillators with time-delayed connections to a phase model. We assume the model for a single oscillator

$$\frac{dX}{dt} = F(X(t)), \tag{1}$$

admits an exponentially asymptotically stable periodic orbit, denoted by $\hat{X}(t)$, with period T. Further, we denote by $Z = \hat{Z}(t)$ the unique periodic solution of the system adjoint to the linearization of (1) about $\hat{X}(t)$ satisfying the normalization condition:

$$\frac{1}{T}\int_0^T \hat{Z}(t)\cdot F(\hat{X}(t))dt = 1.$$

Now consider the following network of identical oscillators with all-to-all, time-delayed coupling

$$\frac{dX_i}{dt} = F(X_i(t)) + \epsilon\sum_{j=1}^{N} w_{ij}G(X_i(t), X_j(t-\tau_{ij})),\ i = 1,\ldots,N. \tag{2}$$

Here $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ describes the coupling between two oscillators, ϵ is referred to as the coupling strength, and $W = [w_{ij}]$ is the coupling matrix. We assume $w_{ij} \geq 0$.

When ϵ is sufficiently small and the w_{ij} are of order 1 with respect to ϵ, we can apply the theory of weakly coupled oscillators to reduce (2) to a phase model [10–12]. The way in which the time delays enter into the phase model depends on the size of the delays relative to other time constants in the model. Let $\Omega = 2\pi/T$. It has been shown [12,46,47] that if the delays satisfy $\Omega\tau_{ij} = O(1)$ with respect to the coupling strength ϵ, then the appropriate model is

$$\frac{d\theta_i}{dt} = \Omega + \epsilon\sum_{j=1}^{N} w_{ij}H(\theta_j - \theta_i - \eta_{ij}) + O(\epsilon^2),\ i = 1, 2,\ldots,N, \tag{3}$$

where $\eta_{ij} = \Omega\tau_{ij}$. That is, the delays enter as phase lags. The interaction function H is a 2π-periodic function which satisfies

$$H(\theta) = \frac{1}{T}\int_0^T \hat{Z}(s)\cdot G(\hat{X}(s), \hat{X}(s+\theta/\Omega))\,ds$$

with $\hat{X}, \hat{Z}$ as defined above.

To study cluster solutions we will make two simplifications. First, we assume that all the delays are equal:

$$\tau_{ij} = \tau,\ \text{i.e.,}\ \eta_{ij} = \eta. \tag{4}$$

Second, we will assume the network has some symmetry. In particular, we will consider the coupling matrix to be in circulant form:

$$W = circ(w_0, w_1, w_2, \ldots, w_{N-1}),\quad \text{equivalently,}\quad w_{ij} = w_{j-i\ (\mathrm{mod}\ N)}. \tag{5}$$

Following [23], we will say the network has connectivity radius r, if $w_k > 0$ for all $k \leq r$, and $w_k = 0$ for all $k > r$. For example, a network with nearest neighbor coupling has connectivity radius $r = 1$. Our results will be derived with the coupling matrix (5), but can be applied to coupling with any connectivity radius by setting the appropriate $w_k = 0$.

We will also assume there is no self coupling, $w_0 = 0$, as this generally the case in applications. The results are essentially unchanged if we include it [49]. These simplifications will apply for the next two sections. In Section 5, we will return to the general model (3).