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Monotonicity, oscillations and stability of a solution to a nonlinear equation modelling the capillary rise

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HIGHLIGHTS

- We prove that the nonlinear equation of the capillary rise has a unique and global solution.
- Small time asymptotics of the solution is found.
- A rigorous proof of the existence of a solution's behaviour transition is given.
- An estimate on the size of the basin of attraction is established.

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ABSTRACT

In this paper we analyse a singular second-order nonlinear ODE which models the capillary rise of a fluid inside a tubular column. We prove global existence, uniqueness and find several approximations along with the asymptotic behaviour of the solution. Moreover, we are able to find a critical value of the nondimensional parameter for which the solution exhibits a transition in its behaviour: from being monotone to oscillatory. This is an analytical rigorous proof of the experimentally and numerically confirmed phenomenon.

1. Introduction

Capillary rise in a narrow vertical tube is a remarkable physical phenomenon that can be observed in a number of everyday situations. One of the most common natural examples of capillary action is water transport in soil or plants. In view of the use of capillary flow in industrial applications, the appropriate mathematical models of its behaviour become increasingly important. One of the most celebrated equations modelling the dynamics of capillary rise was introduced by E.W. Washburn in 1921 (see [1]). His result, the Washburn equation, is a very useful simplification of the liquid rise in the capillary tube. Parallel to the theoretical modelling, many experiments discovered that for certain fluids their free boundary oscillates near the equilibrium (rather that monotonically approach it) [2] and [3].

We can distinguish two dynamical regimes of the fluid flow in a capillary rise experiment. Those regimes correspond to the situations where the fluid column height either increases or oscillates near the Jurin's height (the point, where the capillary pressure balances the weight of a fluid). These situations depend on the ratio of two dimensionless quantities, namely Ohnesorge and Bond numbers. Experiments show (see for ex. [4] and [5]) that there exists a critical value of the dimensionless parameter which separates those both regimes (see [6] and [7]).

In the next sections we investigate the governing equation of the capillary rise phenomenon. It is a singular second-order nonlinear ODE for which we impose mathematically consistent and physically meaningful initial conditions. The problem of posing a non-contradictory starting data seems to be nontrivial and many authors propose different resolutions of this problem (see [8] and [9]). In what follows we prove existence, uniqueness and find the small-time asymptotics for a solution of the considered equation. Moreover, transforming it into an autonomous system we are able to find the exact value of the critical parameter, as well as determine the rate at which the oscillations decay. We conclude this paper with an estimate of the size of the basin of attraction of the considered stationary point.

There is an extensive literature concerning nonlinear oscillations modelled by differential equations. The fundamental monograph [10] contains a collection of mathematical results on dynamical systems and chaos. On the other hand, [11] has a more

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applied flavour. Many authors have contributed to the theory by proving a number of theorems concerning conditions sufficient for oscillatory motion described by ordinary second-order equations (see for ex. [12–14]) and also, their delay [15] and impulsive generalizations [16]. As for the nonlinear world, throughout the last several decades, a number of different classes of oscillations have been thoroughly investigated (notably: chaos theory, which we do not consider here). We would like to mention only a few classical papers from which [17] was one of the first giving oscillations theorems for a particular class of nonlinear equations. These results were further generalized in [18–20]. Some recent advances concern oscillation theorems for delay [21] and neutral equations [22,23], fractional equations [24–26], problems derived from p-Laplacian [27] and analysis on time-scales [15].

2. Model and nondimensionalization

To find the governing equation for the capillary rise inside a tube we have to balance all the forces acting on a fluid column. Assuming that the cross-section of the tube is constant, we can consider only the one-dimensional version of the problem. Forces that take part in this process are of capillary, gravitational and viscous nature. Following [8,28] and [29] we can derive the governing equation as a consequence of the Newton's second law

$$\frac{8\mu}{r^2}hh' + \rho g h + \rho \frac{d}{dt}\left(hh'\right) = \frac{2\gamma\cos\theta}{r}, \tag{1}$$

where $h = h(t)$ is the liquid column height at time t, μ is the viscosity, ρ is the density, γ is the surface tension, θ being the contact angle while r radius of the capillary tube. As usual, g denotes the gravitational acceleration. Going from left to right, the terms in (1) are related to: viscosity (Hagen–Poiseuille flow), hydrostatic pressure, inertia and capillary action (Young–Laplace law) respectively. Further, we introduce the constant h_e which is the so-called Jurin height [3] and is equal to

$$h_e = \frac{2\gamma\cos\theta}{r\rho g}. \tag{2}$$

This is precisely the height at which the capillary force balances the weight of a liquid. As can also be seen from Eq. (1) Jurin's height is the anticipated steady-state solution of the governing equation.

Now we proceed to the non-trivial task of choosing the initial conditions. For the starting value of the liquid height we take

$$h(0) = 0, \tag{3}$$

while, to get the initial value for the velocity we should make a certain assumption guaranteeing the well-posedness. If we rewrite the last term in (1) in the form $\frac{d}{dt}\left(hh'\right) = h'^2 + hh''$, our governing equation becomes

$$\frac{8\mu}{r^2}hh' + \rho g h + \rho(h'^2 + hh'') = \frac{2\gamma\cos\theta}{r}. \tag{4}$$

After putting $t = 0$ we can compute the initial condition for the velocity, namely

$$h'(0) = \sqrt{\frac{2\gamma\cos\theta}{\rho r}}. \tag{5}$$

Notice that assuming vanishing initial height, the above formula is the *only* choice we can make in order to state a consistent problem. In [8] a somewhat different approach has been proposed: the initial velocity was also set to zero. On the other hand, if we substitute this value along with initial condition for the height, a contradiction in (4) will occur.

Another, and more physical, choice of the initial condition is based on an assumption that at the beginning of the experiment the fluid velocity is zero. Then, due to the entrance effect the initial height is proportional to the Jurin's height and is fixed by the tube's radius. Specifically,

$$h(0) = \alpha h_e, \quad h'(0) = 0, \tag{6}$$

for some $0 < \alpha < 1$ dependent on the experimental set-up. A thorough discussion of the appropriate initial conditions for the capillary rise experiment has been given in [5,2,9], where both of the above remedies for the initial singularity of (1) were explained. As will be explicitly noted below, almost all of our results are independent of the specific choice of the initial conditions. In what follows, for the sake of simplicity, we shall use (3) with (5) and remark, where appropriate, how our proofs can be modified to account for (6).

To end our preparations we will cast the model into a dimensionless form. The correct and usual scaling is the following

$$H = \frac{h}{h_e}, \quad T = \frac{t}{\tau}, \tag{7}$$

where h_e is the Jurin's height (2) while τ is the specific time scale to be chosen next. This scaling along with (1) yields

$$\frac{8\mu h_e^2}{r^2\tau}HH' + \rho g H + \frac{\rho h_e^2}{\tau^2}\left(HH'\right)' = P_c, \tag{8}$$

where, as a slight abuse of notation, the prime denotes differentiation with respect to T. Since, in usual situations, the main balance is between the capillary and hydrostatic forces, we take

$$\tau = \frac{8\mu P_c}{\rho^2 g^2 r^2}, \tag{9}$$

which transforms (8) and initial conditions (3) and (5) into

$$HH' + H + \omega\left(HH'\right)' = 1, \tag{10}$$

and the initial conditions

$$H(0) = 0, \quad H'(0) = \frac{1}{\sqrt{\omega}}, \tag{11}$$

for the dimensionless parameter

$$\omega = \frac{\rho^2 g r^4}{64\mu^2 h_e}. \tag{12}$$

3. Analytical results

In this section we will prove our main results concerning estimates, transition of the behaviour and the stability of the solution to (10).

3.1. Existence, uniqueness and estimates

First, we present a technical lemma giving a simplification of the governing equation. It will be useful in the following considerations.

Lemma 1. *The problem (10) with initial conditions (11) can be transformed via $u(s) = \frac{1}{2}H(T)^2$ with $s = T/\sqrt{\omega}$ into*

$$u'' + \frac{1}{\sqrt{\omega}}u' + \sqrt{2u} = 1, \tag{13}$$

with

$$u(0) = 0, \quad u'(0) = 0. \tag{14}$$

Once again, prime denotes a differentiation with respect to the independent variable, which here is taken to be s.