ARTICLE IN PRESS

Physica D ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

4-wave dynamics in kinetic wave turbulence

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HIGHLIGHTS

- We deal with 4-wave Hamiltonian systems in the framework of wave turbulence.
- Averaging technique based on the *Feynman–Wyld* diagrams.
- Kinetic limit : leading order equations for the statistics evolution are derived.
- *Random-phase* and *random-phase and amplitude* properties preserved in time.
- Powerful tool to investigate many relevant physical systems.

ARTICLE INFO

Article history:
Received 30 March 2017
Received in revised form 30 August 2017
Accepted 7 September 2017
Available online xxxx
Communicated by C. Josserand

Keywords:
Weak wave turbulence
Kinetic theory
Intermittency

ABSTRACT

A general Hamiltonian wave system with quartic resonances is considered, in the standard kinetic limit of a continuum of weakly interacting dispersive waves with random phases. The evolution equation for the multimode characteristic function Z is obtained within an "interaction representation" and a perturbation expansion in the small nonlinearity parameter. A frequency renormalization is performed to remove linear terms that do not appear in the 3-wave case. Feynman–Wyld diagrams are used to average over phases, leading to a first order differential evolution equation for Z. A hierarchy of equations, analogous to the Boltzmann hierarchy for low density gases is derived, which preserves in time the property of random phases and amplitudes. This amounts to a general formalism for both the N-mode and the 1-mode PDF equations for 4-wave turbulent systems, suitable for numerical simulations and for investigating intermittency. Some of the main results which are developed here in detail have been tested numerically in a recent work.

1. Introduction

Wave Turbulence (WT) theory concerns the dynamics of dispersive waves that interact nonlinearly over a wide range of scales [1]. In general the nonlinear interaction can be considered small, allowing a perturbative analysis and then an asymptotic closure for statistical observables [2]. For this reason, sometimes one then talks about Weak Wave Turbulence (WWT). Until recently, most of the attention was given to the energy spectrum, which is governed by a kinetic equation. Wave turbulence also provides exact solutions of the kinetic equation, which are related to equipartition, Rayleigh–Jeans solution, or stationary cascade, Kolmogorov–Zakharov solutions [3]. Many physical phenomena are studied within this general framework, for instance gravity [4–7], capillary or Alfvèn waves [8–11], non-linear optics [12] and elastic plates [13–15]. Furthermore, applications of WT to non dispersive systems such as the acoustic waves [16,17] exist, even though the necessary statistical closure is subtler in such cases [18,19].

In the last years, many experiments and numerical simulations were performed to verify the predictions of WT. The picture is relatively clear in the case of the capillary waves on a fluid surface (water, ethanol, liquid hydrogen or liquid helium): both experiments and numerical simulations confirm the Kolmogorov–Zakharov spectrum predicted by WT in this case. For other cases, *e.g.* surface gravity waves or waves in vibrating elastic plates, the picture is more complicated: both numerics and experiments showed deviations from theoretical

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http://dx.doi.org/10.1016/j.physd.2017.09.001
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predictions, and the presence of intermittency [20–23]. This was unexpected, since WT appears as a mean-field theory, based on an initial "quasi-Gaussianity", previously believed to prevent sensible deviations from Gaussianity.

An important step forward in this context has been the development of a more efficient formalism for non-Gaussian wavefields [1,24–26]. In particular, these works pointed out that probability density functions (PDF) are the relevant statistical objects to be analyzed, reviving the interest in the study of PDFs in WT, that dates back to the works of Peierls, Brout, Prigogine, Zaslavskii and Sagdeev [27–29]. These authors had considered waves in anharmonic crystals, which constitute a special case of 3-wave systems. In the recent developments a diagrammatic approach was proposed [1], based on Zakharov's pioneering work [30,31], to analytically investigate PDF equations. Importantly, this has also clarified the role of the different assumptions needed for the statistical closure. In particular, the 3-wave resonant systems has been studied in detail and a Peierls equation for the N-particles PDF has been proposed [1,24,25].

Nevertheless, the Peierls equation does not guarantee the strict preservation of the independence of phases and amplitudes, even though it can be argued that the property of *random phases and amplitudes* (RPA) is preserved in a weaker form [1,32]. Starting from these premises, it has been shown that a proper normalization of the wave amplitudes is necessary for 3-wave resonant systems, in order to obtain a finite spectrum in the infinite-box limit, that leads to an amplitude density, dependent on the continuous variable $\mathbf{k}$ [33]. In particular, the original amplitudes must be normalized by a factor scaling as $1/V$, where V is the volume of the box. Adopting such a point of view, the Peierls equation for the multimode PDFs is not the leading-order asymptotic equation of the continuum limit of weakly interacting, incoherent waves. In Ref. [33], then, new multimode equations were derived, that importantly have the factorized exponential solutions excluded by the Peierls equation. This is equivalent to the preservation of the RPA property. In turn, the preservation of exponential solutions implies a law of large numbers (LLN) for the empirical spectrum at times $\tau > 0$, which is analogous to the propagation of chaos of the BBGKY hierarchy in the kinetic theory of gases. This LLN implies that the empirical spectrum satisfies the wave-kinetic closure equations for nearly every initial realization of random phases and amplitudes, without necessity of averaging. Just as the Boltzmann hierarchy has factorized solutions for factorized initial conditions, so does the kinetic wave hierarchy for all multi-point spectral correlation functions. An H-theorem corresponding to positive entropy variation holds as well. On the other hand, using these multimode equations, Ref. [33] shows that the 1-mode PDF equations are not altered by the different normalization, if the modes initially enjoy the RPA property.

The 4-wave case has not yet been dealt with, although a formal analogy has been used to propose a possible extension of the 3-wave result to the 4-wave case [32]. Therefore, the present paper is devoted to the case of 4-wave interactions, which is of particular interest. As a matter of fact, most of the known violations of Gaussianity arise in gravity waves and in vibrating elastic plates, which are 4-wave resonant systems. Following the same diagrammatic approach of Ref. [1], and using the normalization proposed in Ref. [33], we first explicitly derive the continuous multimode equations, and then we obtain the equation for the M-mode PDF equation. These equations are different from the Peierls equations obtained by the formal analogy of Ref. [32]; they constitute instead a direct extension of the 3-wave case treated in Ref. [33]. The relation between the Peierls and our equations is thus discussed, showing the limit in which they coincide. Our framework also sheds some light on the issue of WT intermittency, as demonstrated by a companion paper [34], in which the equations obtained here are confirmed by numerical simulations of two 4-wave resonant Hamiltonian systems.

This work is organized as follows. First, we describe our model and notation, which are consistent with previous works [1,33]. Section 2 discusses the probabilistic properties of RPA fields. The main results of this paper are reported in Sections 3 and 4, where the multimode equations are derived and discussed. In Section 3 the spectral generating functional and correlation functions are considered, while Section 4 concerns the PDF generating function and the multipoint PDFs. Section 5 summarizes our results. Technical details are provided in Appendix A, Appendix B, and Appendix C, in which we also briefly explain the diagrams used to calculate the averages.

1.1. Model and notation

Similarly to [33], we consider a complex wavefield $u(\mathbf{x}, t)$ in a d-dimensional periodic cube with side L. This field is a linear combination of the canonical coordinates and momenta. It is assumed that there is a maximum wavenumber $k_{\max}$, to avoid ultraviolet divergences. This can be achieved by a lattice regularization with spacing $a = L/M$, for some large integer M, so that $k_{\max} = \pi/a$. The location variable x then ranges over the physical space

$$\Lambda_L = a\mathbb{Z}_M^d, \tag{1}$$

with the usual notation $\mathbb{Z}_M$ for the field of integers, modulo M. This space has volume $V = L^d$. The dual space of wavenumbers is

$$\Lambda_L^* = \frac{2\pi}{L}\mathbb{Z}_M^d \tag{2}$$

with $k_{\min} = 2\pi/L$. The total number of modes is $N = M^d$, so that $V = Na^d$. The following index notation will be used:

$$u^\sigma(\mathbf{x}) = \begin{cases} u(\mathbf{x}) & \sigma = +1 \\ u^*(\mathbf{x}) & \sigma = -1 \end{cases} \tag{3}$$

for u and its complex-conjugate u^*. Likewise, we adopt the convention for (discrete) Fourier transform:

$$A^\sigma(\mathbf{k}) = \frac{1}{N}\sum_{\mathbf{x}\in\Lambda_L} u^\sigma(\mathbf{x}, t)\exp(-i\sigma\mathbf{k}\cdot\mathbf{x}) \tag{4}$$

so that $A^+(\mathbf{k})$ and $A^-(\mathbf{k})$ are complex conjugates. This quantity converges to the continuous Fourier transform $\frac{1}{L^d}\int_{[0,L]^d} d^dx\, u^\sigma(\mathbf{x}, t)$ $\exp(-i\sigma\mathbf{k}\cdot\mathbf{x})$ in the limit $a \to 0$. The discrete inverse transform is

$$u^\sigma(\mathbf{x}) = \sum_{\mathbf{k}\in\Lambda_L^*} A^\sigma(\mathbf{k})\exp(i\sigma\mathbf{k}\cdot\mathbf{x}). \tag{5}$$