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Energy transfer in autoresonant Klein–Gordon chains

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HIGHLIGHTS

- Capture into resonance for the resonant Klein–Gordon chain is studied.
- The threshold values of the structural and excitation parameters are determined.
- An effect of the slow modulation of the external frequency is investigated.
- Explicit asymptotic approximations to the quasi-steady solutions are derived.
- The asymptotic approximations agree with the exact (numerical) results.

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ABSTRACT

In this work we examine autoresonant oscillations in a Klein–Gordon chain of finite length. The chain is subjected to an external periodic forcing with a slowly varying frequency applied at one edge of the chain. Explicit asymptotic equations describing the amplitudes and the phases of oscillations are derived. These equations demonstrate that, in contrast to the chains with linear attachments, the nonlinear chain can be entirely captured into resonance provided that its structural and excitation parameters exceed their critical thresholds. It is shown that at large times the amplitudes of AR oscillations converge to a monotonically growing mean amplitude that is equal for all oscillators. The threshold values of the structural and excitation parameters, which allow the emergence of autoresonance in the entire chain, are determined. The derived analytic results are in good agreement with numerical simulation.

1. Introduction

In this work we investigate the emergence of high-energy autoresonant (AR) oscillations in a Klein–Gordon chain of identical linearly coupled Duffing oscillators. The chain excited by a harmonic force with a slowly varying frequency applied at an edge of the chain.

An idea of autoresonance, or "resonance under the action of a force produced by the system's itself" was first suggested by Andronov, Vitt and Khaikin [1]. In particular, it was shown that the proper modulation of structural and/or excitation parameters may lead to the occurrence of AR in a nonlinear oscillator. After first studies for the purposes of particle acceleration [2–4], a large number of theoretical approaches, experimental results and applications of AR in different fields of natural science, from plasmas to nonlinear optics and hydrodynamics, have been reported in literature (see, e.g., [5–7] and references therein). The analysis was first concentrated on the study of AR in a single nonlinear oscillator but then it was extended to the systems with two- or three-degree-of freedom (2DOF or 3DOF) systems. Examples in this category are excitations of continuously phase-locked plasma waves with laser beams [8], particle transport in a weak external field with slowly changing frequency [9], isotope separation process [10], control of nanoparticles [11], etc. It is important to note that multidimensional nonlinear nonstationary systems seldom yield the explicit analytical solutions needed for understanding and modeling the transition phenomena, so that these studies have not suggested any general conclusions concerning the occurrence of AR in multidimensional systems.

In most of previous studies, AR in the forced oscillator was considered as an effective tool for exciting high-energy oscillations in the entire array. However, recent results [12,13] have shown that this principle is not universal because capture into resonance of a multi-particle chain is a much more complicated phenomenon than a similar effect for a single oscillator [14] and the behavior of each element in the chain may differ from the dynamics of a single oscillator. This effect was recently analyzed for oscillator arrays, which comprise a chain of time-invariant linear oscillators weakly coupled to a nonlinear actuator [12,13]. It was shown that external periodic forcing with slowly-varying frequency gives rise to AR only in the excited nonlinear oscillator (the actuator), while the response of the linear attachment remains bounded.

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This work demonstrates that, in contrast to the above-mentioned examples, the nonlinear chain can be entirely captured into resonance. The difference in the dynamics of these two types of systems is closely connected with their resonance properties. High-energy resonant oscillations in a linear time-invariant oscillator are generated by an external force, whose frequency is equal or close to the natural frequency of the oscillator but deviations of the forcing frequency result in escape from resonance. On the contrary, the natural frequency of a nonlinear oscillator changes as its amplitude changes, so that the oscillator may stay captured into resonance with its drive if the driving frequency vary slowly in time to be consistent with the slowly changing frequency of the oscillator. The ability of a nonlinear oscillator to stay captured into resonance due to variance of its structural or excitation parameters is termed *autoresonance* (*AR*), or nonstationary resonance.

It was shown in recent papers that the emergence and stability of AR in a single Duffing oscillator [14] as well as in two coupled oscillators [15] directly depend on the forcing and coupling parameters. It was demonstrated both theoretically and numerically that AR can occur only if the considered parameters exceed a certain threshold. Our consideration of a multi-particle chain is also focused on the study of the threshold phenomena, with the purpose to identify a set of parameters allowing stable AR in the entire chain.

In this paper, the analysis of AR is motivated by the results earlier obtained in the study of a Klein–Gordon chain subjected to periodic forcing with constant frequency [16]. Section 2 introduces the equations of the chain dynamics. As in the case of periodic excitation [16], the small parameter of the system is defined as dimensionless frequency detuning. The procedure of averaging yields a set of equations for the slow envelopes and phases of resonant oscillations. In Section 3 we analytically calculate the steady-state amplitudes and phases of oscillations. Both analytical results and numerical simulations demonstrate that in the main approximations all amplitudes converge to a slowly increasing quasi-steady value equal for all oscillators thus demonstrating equipartition of energy at large times. In Section 4 the critical thresholds for the structural and excitation parameters are derived. Numerical examples are discussed in Section 5.

2. The model

The chain dynamics is governed by the following equations:

$$\begin{aligned}
&\frac{d^2u_1}{dt^2}+\omega^2u_1+\gamma u_r^3+\kappa(u_1-u_2)=A\cos\theta(t),\\
&\frac{d^2u_r}{dt^2}+\omega^2u_r+\gamma u_r^3+\kappa(2u_r-u_{r+1}-u_{r-1})=0,\\
&\quad r=2,\ldots,n-1,\\
&\frac{d^2u_n}{dt^2}+\omega^2u_n+\gamma u_r^3+\kappa(u_n-u_{n-1})=0,\\
&\frac{d\theta}{dt}=\omega+\zeta(t),\ \zeta(t)=k_1+k_2t.
\end{aligned}\tag{2.1}$$

Here and below, the variable u_r denotes the position of the rth oscillator; $\omega^2=c/m$, m and c being the mass and the linear stiffness of each oscillator; $\gamma>0$ is the cubic stiffness coefficient; the coefficient κ represents the stiffness of linear coupling between the oscillators. The first oscillator is subjected to periodic forcing with amplitude A and time-dependent frequency $\Omega(t)=\omega+\zeta(t)$, where $\zeta(t)=k_1+k_2t$; the parameters $k_1>0$ and $k_2>0$ denote initial constant detuning and detuning rate, respectively. Note that the array (2.1) may be considered as an example of a microelectromechanical system (MEMS) with a broad spectrum of applications (see, e.g., [17] and references therein).

We reduce (2.1) to the form more convenient for further analysis. First, assuming small initial frequency detuning, we introduce the small parameter of the system $\varepsilon=k_1/\omega$, $0<\varepsilon\ll 1$. Then, taking into consideration the resonance properties of the oscillators, we introduce the following rescaled parameters:

$$\begin{aligned}
&\tau_0=\omega t,\ \tau=\varepsilon\tau_0,\\
&\gamma=8\varepsilon\alpha\omega^2,\ A=2\varepsilon F\omega^2,\ \kappa=2\varepsilon k\omega^2,\ k_2=2\varepsilon^2\beta\omega^2.
\end{aligned}\tag{2.2}$$

In these notations, the equations of motion take the form

$$\begin{aligned}
&\frac{d^2u_1}{d\tau_0^2}+u_1+8\varepsilon\alpha u_1^3+2\varepsilon k(u_1-u_2)=2\varepsilon F\sin\theta(\tau_0,\varepsilon),\\
&\frac{d^2u_r}{dt^2}+u_r+8\varepsilon\alpha u_r^3+2\varepsilon k(2u_r-u_{r+1}-u_{r-1})=0,\\
&\quad r=2,\ldots,n-1,\\
&\frac{d^2u_n}{d\tau_0^2}+u_n+8\varepsilon\alpha u_n^3+2\varepsilon k(u_n-u_{n-1})=0,\\
&\frac{d\theta}{d\tau_0}=1+\varepsilon\zeta_0(\tau),\ \zeta_0(\tau)=1+\beta\tau.
\end{aligned}\tag{2.3}$$

The system is assumed to be initially at rest, i.e. $\theta=0$, $u_r=0$, $v_r=\frac{du_r}{d\tau_0}=0$ at $\tau_0=0$ $(r=1,\ldots,n)$. We recall that zero initial conditions identify the so-called *Limiting Phase Trajectories* (LPTs) corresponding to maximum possible energy transfer from a source of energy to a receiver (see, e.g., [18,19] and references therein).

Asymptotic solutions of Eqs. (2.3) for small ε are derived with the help of the multiple time scale formalism [20]. First, we introduce the dimensionless complex-conjugate vector envelopes Ψ and Ψ^* with components Ψ_r,Ψ_r^*, $r=1,\ldots,n$, and related dimensionless parameters by formulas

$$\begin{aligned}
&\Psi_r=\Lambda^{-1}(v_r+iu_r)e^{-i\theta},\ \Psi_r^*=\Lambda^{-1}(v_r-iu_r)e^{i\theta},\\
&\Lambda=(1/3\alpha)^{1/2},\ f=F/\Lambda,\ \mu=k/\Lambda.
\end{aligned}\tag{2.4}$$

It follows from (2.4) that the real-valued dimensionless amplitudes and phases of oscillations are given by $\widetilde{a}_r=|\Psi_r|$ and $\widetilde{\Delta}_r=\arg\Psi_r$, respectively.

Substituting (2.4) into (2.3), we obtain the following equations for the envelopes Ψ_r:

$$\begin{aligned}
&\frac{d\Psi_1}{d\tau_0}=-i\varepsilon[(\zeta_0(\tau)-|\Psi_1|^2)\Psi_1-\mu(\Psi_1-\Psi_2)+f+G_1],\\
&\frac{d\Psi_r}{d\tau_0}=-i\varepsilon[(\zeta_0(\tau)-|\Psi_r|^2)\psi_r-\mu(2\Psi_r-\Psi_{r-1}\\
&\quad-\Psi_{r+1})+G_r],\ \ 2\le r\le n-1,\\
&\frac{d\Psi_n}{d\tau_0}=-i\varepsilon[(\zeta_0(\tau)-|\Psi_n|^2)\Psi_n-\mu(\Psi_n-\Psi_{n-1})+G_n],
\end{aligned}\tag{2.5}$$

with initial conditions $\Psi_r(0)=0$, $r=1,\ldots,n$. The coefficients G_r include fast harmonics with coefficients depending on Ψ and Ψ^* but an explicit expression of these coefficients is insignificant for further analysis.

In order to construct asymptotic approximations to the solutions of (2.5), we employ the multiple time scale approach [20]. First, the following asymptotic decomposition of the variables Ψ_r is introduced:

$$\Psi_r(\tau_0,\tau,\varepsilon)=\psi_r(\tau)+\varepsilon\psi_r^{(1)}(\tau_0,\tau)+O(\varepsilon^2),\ r=1,\ldots,n,\tag{2.6}$$

with the leading-order terms $\psi_r(\tau)$ being the slow envelopes. The equations for $\psi_r(\tau)$ can be obtained by straightforward averaging