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Dispersive shock waves in the Kadomtsev–Petviashvili and two dimensional Benjamin–Ono equations

Mark J. Ablowitz [a], Ali Demirci [a,b,*], Yi-Ping Ma [a]

[a] Department of Applied Mathematics, University of Colorado, 526 UCB, Boulder, CO 80309-0526, USA

[b] Department of Mathematics, Faculty of Science and Letters, Istanbul Technical University, Istanbul 34469, Turkey

HIGHLIGHTS

- DSWs in KP and 2DBO equations are considered for parabolic front initial data.
- KP and 2DBO equations are reduced to cKdV and cBO equations.
- Whitham modulation systems for cKdV and cBO equations are derived.
- Numerics of Whitham systems are compared with numerics of cKdV/cBO equations.
- Numerics of KP/2DBO equations are compared with numerics of cKdV/cBO equations.

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ABSTRACT

Dispersive shock waves (DSWs) in the Kadomtsev–Petviashvili (KP) equation and two dimensional Benjamin–Ono (2DBO) equation are considered using step like initial data along a parabolic front. Employing a parabolic similarity reduction exactly reduces the study of such DSWs in two space one time $(2+1)$ dimensions to finding DSW solutions of $(1+1)$ dimensional equations. With this ansatz, the KP and 2DBO equations can be exactly reduced to the cylindrical Korteweg–de Vries (cKdV) and cylindrical Benjamin–Ono (cBO) equations, respectively. Whitham modulation equations which describe DSW evolution in the cKdV and cBO equations are derived and Riemann type variables are introduced. DSWs obtained from the numerical solutions of the corresponding Whitham systems and direct numerical simulations of the cKdV and cBO equations are compared with very good agreement obtained. In turn, DSWs obtained from direct numerical simulations of the KP and 2DBO equations are compared with the cKdV and cBO equations, again with good agreement. It is concluded that the $(2+1)$ DSW behavior along self similar parabolic fronts can be effectively described by the DSW solutions of the reduced $(1+1)$ dimensional equations.

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1. Introduction

In recent years the study of dispersive shock waves (DSWs) has generated considerable interest. In water waves DSWs have also been termed undular bores [1,2]. In fact, an early observation of an undular bore goes back to 1850 [3]. In plasma physics a careful observation of a DSW, sometimes referred to as a collisionless shock wave, was made over 40 years ago [4]. More recent experiments/observations of DSWs have been carried out in other fields, e.g. Bose–Einstein condensates (BEC) [5,6] and nonlinear optics [7–9]. Mathematically speaking the study of DSWs is difficult since the profile of the shock wave is highly oscillatory and the underlying shock solution does not converge strongly. A prototypical example of a DSW occurs in the KdV equation

$$u_t + uu_x + \epsilon^2 u_{xxx} = 0 \tag{1.1}$$

with $\epsilon^2 \ll 1$ and initial conditions corresponding to a simple unit step (Heaviside) function. In 1974, employing an averaging method pioneered by Whitham [10], Gurevich and Pitaevskii [11] gave a detailed description of the associated DSW. About 10 years later Lax and Levermore [12] described the DSW rigorously via inverse scattering transform methods. Over the years there have been numerous important analytical studies that employ Whitham methods cf. [11,13–16].

* Corresponding author at: Department of Mathematics, Faculty of Science and Letters, Istanbul Technical University, Istanbul 34469, Turkey.
E-mail addresses: mark.ablowitz@colorado.edu (M.J. Ablowitz), demircial@itu.edu.tr (A. Demirci), yiping.m@gmail.com (Y.-P. Ma).

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Here we study two space one time (2 + 1) dimensional equations, including the Kadomtsev–Petviashvili (KP) [17] and the two dimensional Benjamin–Ono (2DBO) [18] equations (see Eqs. (2.1) and (2.2)), by employing a parabolic similarity reduction and thereby exactly reducing these equations to the (1 + 1)-dimensional cylindrical KdV (cKdV) and cylindrical Benjamin–Ono (cBO) equations (see Eqs. (2.19) and (2.20)) respectively. We note that 2 + 1 dimensional NLS type equations and associated DSW solutions were analyzed by reducing them to associated 1 + 1 dimensional systems [19,20]. The goal of this paper is to shed light on KP dynamics when there is step like data along parabolic front in the initial data. Step like initial data is often termed as a Riemann problem in shock wave studies. Implications of the importance of parabolic type fronts in the KP equations can be found in Refs [21,22]. As mentioned above, it is noteworthy that applications to multidimensional shallow water bores seem to have similar structure [23]. More general initial data given along such fronts will require a more general approach to Whitham theory; this analysis is outside the scope of this paper.

We analyze the cKdV/cBO equations via Whitham theory and derive Whitham modulation equations; these equations are transformed into simpler form by introducing appropriate Riemann type variables. These Whitham equations in Riemann variables are not in diagonal hydrodynamic (i.e. diagonal conservation system [24]) form. We remark that in the cKdV case a diagonal hydrodynamic form may be obtained using the integrability of cKdV [25–27]; on the other hand, neither 2DBO nor its reduction, the cBO equation is known to be integrable.

We study the DSWs in the cKdV and cBO equations numerically and describe their differences from the DSWs in the classical KdV and BO equations. Indeed the DSWs in the former are found to decay slowly in time whereas those in the latter do not exhibit such temporal decay. We find that direct numerical simulations of the Whitham modulation equations agree well with those of the cKdV and cBO equations.

We then compare these (1 + 1) dimensional DSW structures to direct numerical simulations of the (2 + 1) dimensional KP and 2DBO equations. After fixing parameters our comparisons between 1+1 numerics/theory and 2+1 numerics exhibit very good results. In general the DSW weakens across the parabolic front as time increases. We also note that the numerical simulations of 1 + 1 Whitham theory which removes the fast variation, are much faster than the 1 + 1 cKdV/cBO equations which in turn, are orders of magnitude faster than the 2 + 1 equations.

Over the years there have been many numerical studies and calculations associated with the KP equation cf. [28–32].

Our interest is to study DSW systems which have step like data across a parabolic front; this is analogous to the well known Riemann or shock tube problem in classical shock waves. We find that indeed there are DSWs generated across the shock front. To our knowledge this is the first time the nondecaying 2+1 analogue of a Riemann problem for KP and 2DBO is analyzed in detail.

The reduction discussed here, which we also term parabolic front tracking, was used [33,34] in the analysis of the Khokhlov–Zabolotskaya (KZ) equation [35] (see also [36]). Indeed the KP/2DBO equation in the limit of $\epsilon \to 0$ (zero dispersion in the x direction) reduces to the KZ equation. When viscosity is added to the KZ equation the relevant shock waves are strongly convergent. Our study of the KP/2DBO DSWs requires critical use of Whitham modulation theory, which is necessary due to the weak convergence of the DSWs. In the context of 2+1 dispersive systems connections to cylindrical systems such as cKdV was also found [37] (cf. [38] and refs included).

This paper is organized as follows. In Section 2 a parabolic similarity reduction is used to exactly transform the KP and 2DBO equations to the cKdV and cBO equations along a parabolic front. In Section 3 we employ perturbation theory [39] to find the conservation laws associated with Whitham theory for the KdV and cKdV equations. We then transform the Whitham modulation equations employing Riemann-type variables; the resulting Whitham system is not immediately in diagonal hydrodynamic form. We solve the 1+1 Whitham system associated with the KdV and cKdV equations numerically and reconstruct the DSW solutions of KdV and cKdV. We then compare these results with direct numerical simulations of KdV and cKdV and show that, apart from an unimportant phase they are in very good agreement. We also note that the Whitham equations for cKdV exhibit a small discontinuity. This discontinuity would be resolved by taking into account higher order terms (see [40]), but doing so is outside the scope of this paper. In Section 4 the BO and cBO equations are analyzed in the same way as KdV and cKdV are analyzed in Section 3. In Section 5 we compare the 1 + 1 results for cKdV/cBO and the 2 + 1 results for KP/2DBO by direct numerical simulations (see also [41,42] for numerics associated with KP). After accounting for an unimportant mean term we again find excellent agreement; animations are also included as part of our 2 + 1 description. Numerical implementation of the 2+1 KP/2DBO equations employ (regularized) step-like data along a parabolic front; to avoid boundary interactions the front is taken to decay at large distances. The results in a region around the x-axis, consistent with the eventual decay, approximate well the parabolic front. We conclude in Section 6.

2. Reduction of KP, 2DBO equations to cKdV, cBO equations

In this section, we examine DSW propagation associated with two different (2 + 1) dimensional nonlinear partial differential equations (PDEs). One is the Kadomtsev–Petviashvili (KP) equation

$$\left(u_t + uu_x + \epsilon^2 u_{xxx}\right)_x + \lambda u_{yy} = 0 \tag{2.1}$$

where ϵ, λ are constants. This equation was first derived by Kadomtsev–Petviashvili (KP) [17] in the context of analyzing the stability of the KdV soliton in a 2 + 1 setting subject to weak transverse variations; subsequently it was derived in water waves [43] where it describes the evolution of weakly nonlinear two dimensional long water waves of small amplitude. When $|\epsilon| \ll 1$ we have weak dispersion. According to the sign of λ, Eq. (2.1) is usually termed *KP-I* $(-)$ or *KP-II* $(+)$, respectively. KP-I describes the dynamics when the surface tension of the water is strong and KP-II describes the dynamics with weak surface tension. The other equation we study is

$$\left(u_t + uu_x + \epsilon \mathcal{H}\left(u_{xx}\right)\right)_x + \lambda u_{yy} = 0 \tag{2.2}$$

where $\mathcal{H}u(x)$ denotes the Hilbert transform:

$$\mathcal{H}u(x) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{u(x')}{x'-x}dx' \tag{2.3}$$

and $\mathcal{P}$ denotes the Cauchy principal value. We refer to Eq. (2.2) as the 2DBO (Two Dimensional Benjamin–Ono) equation; it is a two-dimensional extension of the classical BO equation and describes weakly nonlinear long internal waves in fluids of great depth [18].

The goal in this paper is to enhance understanding of DSWs in multidimensional systems. A general form for these two equations is

$$\left(u_t + uu_x + F_i(u)\right)_x + \lambda u_{yy} = 0; \tag{2.4}$$

when $F_1(u) = \epsilon^2 u_{xxx}$ Eq. (2.4) is the KP equation and when $F_2(u) = \epsilon \mathcal{H}\left(u_{xx}\right)$ it is the 2DBO equation. As an evolution Eq. (2.4) can be written as

$$u_t + uu_x + F_i[u] + \lambda \partial_x^{-1} u_{yy} = 0 \tag{2.5}$$

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