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Whitham modulation equations, coalescing characteristics, and dispersive Boussinesq dynamics

Daniel J. Ratliff*, Thomas J. Bridges

Department of Mathematics, University of Surrey, Guildford GU2 7XH, UK

HIGHLIGHTS

- A theory for Whitham modulation with degenerate characteristics is developed.
- A novel method to obtain a universal two-way Boussinesq equation is presented.
- A new emergence of this Boussinesq equation from a complex Klein–Gordon model.
- The method provides insight into the Kelvin–Helmholtz instability.
- The method implies two-way Boussinesq model for water waves is invalid.

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ABSTRACT

Whitham modulation theory with degeneracy in wave action is considered. The case where all components of the wave action conservation law, when evaluated on a family of periodic travelling waves, have vanishing derivative with respect to wavenumber is considered. It is shown that Whitham modulation equations morph, on a slower time scale, into the two way Boussinesq equation. Both the $1+1$ and $2+1$ cases are considered. The resulting Boussinesq equation arises in a universal form, in that the coefficients are determined from the abstract properties of the Lagrangian and do not depend on particular equations. One curious by-product of the analysis is that the theory can be used to confirm that the two-way Boussinesq equation is not a valid model in shallow water hydrodynamics. Modulation of nonlinear travelling waves of the complex Klein–Gordon equation is used to illustrate the theory.

1. Introduction

The Whitham modulation equations are a hyperbolic or elliptic first order quasilinear system of partial differential equations that can arise from the modulation of a family of periodic wavetrains [1,2]. In the simplest $1+1$ setting, with just conservation of wave action and conservation of waves, the Whitham modulation equations (WMEs) are a pair of partial differential equations (PDEs) with two characteristics. In almost all theory and application of Whitham modulation theory these two characteristics are assumed to be distinct. In [3,4] the case where one of the characteristics is zero was considered, and it was found that a rescaling and nonlinear unfolding generates dispersion in the WMEs and the conservation of wave action morphs into the Korteweg–de-Vries (KdV) equation. In this paper, the case of a double characteristic at zero is considered. With a rescaling, we find that the WMEs in this case morph into the two-way Boussinesq equation. To show this, the strategy is to start with a Lagrangian formulation, average, and modulate. However when key derivatives with respect to wavenumber or frequency of the wave action and wave action flux are zero, higher derivatives are required and a slower time scale is needed. This combination changes the modulation equations from the dispersionless WMEs to the two-way Boussinesq equation which has dispersion. This result gives a new mechanism for the emergence of the two-way Boussinesq equation as a model PDE and, since the basic state can be a finite amplitude travelling wave or more generally a relative equilibrium, this observation increases the range of contexts in which it can appear.

The $1+1$ two-way Boussinesq equation in standard form is

$$u_{tt} + \left(u^2 \pm u_{xx}\right)_{xx} = 0, \tag{1.1}$$

for the scalar-valued function $u(x,t)$, with the plus (minus) sign corresponding to the good (bad) versions. This classification indicates whether the dispersion in the linear part, $u_{tt} \pm u_{xxxx}$, generates a well-posed (good) or ill-posed (bad) linear partial differential equation.

* Corresponding author. Tel.: +44 01483 683024.
E-mail address: d.ratliff@surrey.ac.uk (D.J. Ratliff).

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The first appearance of the Boussinesq equation was in equation (26) on page 75 of Boussinesq [5], where it was proposed as a model for water waves, and it has been extensively studied since: it has solitary wave solutions, blow-up for some initial data, and global existence for other initial data, and it is known to be completely integrable (e.g. [6–10]).

The Whitham modulation theory [1,2] starts with a Lagrangian formulation of field equations for some vector of unknowns, say $U(x,t)$,

$$\mathscr{L}(U) = \iint \mathcal{L}(U_t, U_x, U)\,\mathrm{d}x\mathrm{d}t, \qquad (1.2)$$

where for simplicity of discussion the field variables are restricted to (x,t). Suppose there exists a periodic travelling wave solution

$$U(x,t) = \widehat{U}(\theta), \qquad \widehat{U}(\theta+2\pi) = \widehat{U}(\theta), \\ \theta = kx + \omega t + \theta_0, \qquad (1.3)$$

of wavenumber k and frequency ω, where θ_0 is an arbitrary constant phase shift. Average the Lagrangian over θ

$$\mathscr{L}(\omega,k) = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{L}(\omega\widehat{U}_\theta, k\widehat{U}_\theta, \widehat{U})\,\mathrm{d}\theta. \qquad (1.4)$$

Now suppose ω and k are slowly varying in some sense and can be expressed in terms of the variance of the phase,

$$k = \theta_x \quad \text{and} \quad \omega = \theta_t. \qquad (1.5)$$

Combining these two equations gives the *conservation of waves* $k_t = \omega_x$. Substituting (1.5) into (1.4) and taking variations with respect to θ of the averaged Lagrangian gives *conservation of wave action* $\mathscr{A}_t + \mathscr{B}_x = 0$ with $\mathscr{A} = \mathscr{L}_\omega$ and $\mathscr{B} = \mathscr{L}_k$. Combining conservation of waves with conservation of wave action then gives the WMEs

$$k_t = \omega_x \quad \text{and} \quad \mathscr{A}(\omega,k)_t + \mathscr{B}(\omega,k)_x = 0, \qquad (1.6)$$

for the modulation variables (ω, k). The definition of wave action $\mathscr{A}$ and wave action flux $\mathscr{B}$ and the conservation of waves equation are all exact, but their validity depends on some approximation. Using multiple scales in the Whitham theory [11], slow time and space scales are introduced, $T = \varepsilon t$ and $X = \varepsilon x$, and (ω, k) are taken in a neighborhood of some fixed travelling wave parameters,

$$\omega \mapsto \omega + \varepsilon\Omega(X,T,\varepsilon) \quad \text{and} \quad k \mapsto k + \varepsilon q(X,T,\varepsilon).$$

Then to leading order, $q := q(X,T,0)$ and $\Omega := \Omega(X,T,0)$ satisfy

$$q_T = \Omega_X \quad \text{and} \quad \mathscr{A}_\omega\Omega_T + \mathscr{A}_k q_T + \mathscr{B}_\omega\Omega_X + \mathscr{B}_k q_X = 0. \qquad (1.7)$$

With (ω, k) fixed, this equation is linear and, with the assumption $\mathscr{A}_\omega \neq 0$, it can be written in the standard form

$$\begin{pmatrix} q \\ \Omega \end{pmatrix}_T + \mathbf{A}(\omega,k)\begin{pmatrix} q \\ \Omega \end{pmatrix}_X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad (1.8)$$

with

$$\mathbf{A}(\omega,k) = \frac{1}{\mathscr{A}_\omega}\begin{bmatrix} 0 & -\mathscr{A}_\omega \\ \mathscr{B}_k & \mathscr{A}_k + \mathscr{B}_\omega \end{bmatrix}. \qquad (1.9)$$

The WMEs are *hyperbolic* if the eigenvalues of $\mathbf{A}(\omega,k)$ are real and *elliptic* if the eigenvalues of $\mathbf{A}(\omega,k)$ are complex. Ellipticity is an indication that the basic state is unstable to long wave modulational instability [2].

The roots of $\det[\lambda\mathbf{I} - \mathbf{A}(\omega,k)] = 0$ satisfy

$$\lambda = \frac{\mathscr{A}_k + \mathscr{B}_\omega}{2\mathscr{A}_\omega} \pm \frac{1}{\mathscr{A}_\omega}\sqrt{-\Delta}, \qquad \Delta = \det\begin{bmatrix} \mathscr{A}_\omega & \mathscr{A}_k \\ \mathscr{B}_\omega & \mathscr{B}_k \end{bmatrix} \\ = \det\begin{bmatrix} \mathscr{L}_{\omega\omega} & \mathscr{L}_{\omega k} \\ \mathscr{L}_{k\omega} & \mathscr{L}_{kk} \end{bmatrix}.$$

This form for the eigenvalues highlights the Lighthill condition [12]: when $\Delta > 0$ the WMEs are elliptic and when $\Delta < 0$ the WMEs are hyperbolic.

The WMEs have been generalized in several directions: proper nonlinear WMEs, multiphase wavetrains, additional conservation laws, coupling with mean flow, extension to 2 + 1, and inclusion of dissipation, but attention here is restricted to the basic 1+1 single-phase conservative case to capture the nature of the singularity-induced appearance of the classical Boussinesq equation. The theory is then generalized to the 2 + 1 case which will lead to the WMEs morphing into the 2 + 1 Boussinesq equation.

There has been a largely independent development of modulation theory for periodic solutions of non-conservative systems (Howard–Kopell, Burgers' equation, Cross–Newell theory, Kuramoto–Sivashinsky, phase-diffusion equation, etc.). Conservation of waves is still operational in the non-conservative case, but conservation of wave action no longer exists in general, and new scaling is often needed, leading to a different class of modulation equations. For example, the generic phase equation for modulation of periodic wavetrains of reaction–diffusion equations is Burgers' equation (e.g. Doelman et al. [13]) which requires the scaling with $T = \varepsilon^2 t$ and $X = \varepsilon x$.

Here we are interested in the case where the Whitham equations break down via a double zero eigenvalue of $\mathbf{A}(\omega,k)$ in (1.8)–(1.9). We will also find that an additional natural requirement for the Boussinesq equation to emerge is that the double zero eigenvalue of $\mathbf{A}(\omega,k)$ is non-semisimple.

Because of the special form of $\mathbf{A}(\omega,k)$ in (1.9) a double non-semisimple zero eigenvalue can arise in one of two ways

$$\mathscr{A}_\omega \neq 0 \quad \text{but} \quad \mathscr{A}_k = \mathscr{B}_k = 0, \qquad (1.10)$$

or

$$\mathscr{B}_k \neq 0 \quad \text{but} \quad \mathscr{A}_\omega = \mathscr{B}_\omega = 0. \qquad (1.11)$$

The main result of this paper is that in the neighborhood of the first singularity (1.10), the conservation of wave action morphs into the following form of the two-way Boussinesq equation

$$q_T = \Omega_X \\ \mathscr{A}_\omega\Omega_T + \mathscr{B}_{kk}qq_X + \mathscr{K}q_{XXX} = 0. \qquad (1.12)$$

The first equation is just conservation of waves, and the second equation is the conservation of wave action. Combining the two equations in (1.12) gives the two-way Boussinesq equation in universal form

$$\mathscr{A}_\omega q_{TT} + \left(\frac{1}{2}\mathscr{B}_{kk}q^2 + \mathscr{K}q_{XX}\right)_{XX} = 0, \qquad (1.13)$$

which we call the q-Boussinesq equation. Eq. (1.13) is universal in the sense that the coefficients are determined from abstract properties of a Lagrangian, and do not depend on particular equations.

The second condition in (1.11) leads to

$$\Omega_X = q_T \\ \mathscr{B}_k q_X + \mathscr{A}_{\omega\omega}\Omega\Omega_T + \mathscr{M}\Omega_{TTT} = 0. \qquad (1.14)$$

Combining the two equations in (1.14) gives

$$\mathscr{B}_k\Omega_{XX} + \left(\frac{1}{2}\mathscr{A}_{\omega\omega}\Omega^2 + \mathscr{M}\Omega_{TT}\right)_{TT} = 0, \qquad (1.15)$$

which we call the Ω-Boussinesq equation. The coefficients $\mathscr{K}$ and $\mathscr{M}$ are determined by a Jordan chain argument. Examples are given in Sections 7 and 8. The scaling in the first equation (1.13) is $X = \varepsilon x$ and $T = \varepsilon^2 t$ with scaling $T = \varepsilon t$ and $X = \varepsilon^2 x$ in the second equation (1.15).

The theory extends in a natural way to the 2 + 1 case. Conservation of wave action has three components $(\mathscr{A}, \mathscr{B}, \mathscr{C})$, conservation of waves is vector valued, and θ in (1.3) is replaced by