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The propagation of internal undular bores over variable topography

R. Grimshaw *, C. Yuan

Department of Mathematics, University College London, UK

HIGHLIGHTS

- Whitham modulation equations for a variable-coefficient KdV equation.
- Description of an undular bore in a variable medium.
- Simulation of an undular bore passing through a critical point where there is a polarity change.

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ABSTRACT

In the coastal ocean, large amplitude, horizontally propagating internal wave trains are commonly observed. These are long nonlinear waves and can be modelled by equations of the Korteweg–de Vries type. Typically they occur in regions of variable bottom topography when the variable-coefficient Korteweg–de Vries equation is an appropriate model. Of special interest is the situation when the coefficient of the quadratic nonlinear term changes sign at a certain critical point. This case has been widely studied for a solitary wave, which is extinguished at the critical point and replaced by a train of solitary waves of the opposite polarity to the incident wave, riding on a pedestal of the original polarity. Here we examine the same situation for an undular bore, represented by a modulated periodic wave train. Numerical simulations and some asymptotic analysis based on Whitham modulation equations show that the leading solitary waves in the undular bore are destroyed and replaced by a developing rarefaction wave supporting emerging solitary waves of the opposite polarity. In contrast the rear of the undular bore emerges with the same shape, but with reduced wave amplitudes, a shorter overall length scale and moves more slowly.

1. Variable-coefficient Korteweg–de Vries equation

Large amplitude internal wave trains are commonly observed in the coastal ocean, see the reviews by Grimshaw [1], Holloway et al. [2], Ostrovsky and Stepanyants [3], Helfrich and Melville [4], Grimshaw [5], Grimshaw et al. [6] and the book by Vlasenko et al. [7]. Since these are long nonlinear waves it is now widely accepted that the basic paradigm for these waves is based on the Korteweg–de Vries (KdV) equation, first derived in this context by Benney [8] and Benjamin [9] and subsequently by many others, see the aforementioned references. In the usual physical variables to describe internal waves in the coastal ocean the KdV equation is, see the afore-mentioned references,

$$A_t + cA_x + \mu AA_x + \lambda A_{xxx} = 0. \tag{1}$$

Here $A(x, t)$ is the amplitude of the modal function $\phi(z)$, defined by

$$\{\rho_0(c-u_0)^2\phi_z\}_z + \rho_0 N^2\phi = 0, \quad \text{for } -h < z < 0, \tag{2}$$

$$\phi = 0 \quad \text{at } z = -h, \qquad (c-u_0)^2\phi_z = g\phi \quad \text{at } z = 0. \tag{3}$$

This also serves to define the phase speed c. Here $\rho_0(z)$ is the background density field, stably stratified so that $\rho_0 N^2 = -g\rho_{0z} > 0$, $u_0(z)$ is a background horizontal current and h is the undisturbed fluid depth. The coefficients μ, λ are given by

$$I\mu = 3\int_{-h}^{0} \rho_0\,(c-u_0)^2\,\phi_z^3\,dz, \tag{4}$$

$$I\lambda = \int_{-h}^{0} \rho_0\,(c-u_0)^2\,\phi^2\,dz, \tag{5}$$

$$I = 2\int_{-h}^{0} \rho_0\,(c-u_0)\,\phi_z^2\,dz. \tag{6}$$

It is well-known that the KdV equation (1) is integrable, and its principal solution is the solitary wave, as the outcome of a localised

* Corresponding author. Tel.: +44 20 7679 2856.
E-mail address: r.grimshaw@ucl.ac.uk (R. Grimshaw).

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initial condition is a finite set of rank-ordered solitary waves and some small-amplitude dispersing radiation, see Whitham [10]; Ablowitz and Segur [11]. Here we are concerned with the undular bore solution, which can be found as the outcome of a step initial condition by using the Whitham modulation equations, see Gurevich and Pitaevskii [12].

When the depth h, and background current u_0 and density ρ_0 vary slowly in the horizontal direction with x, the KdV equation (1) is replaced by a variable-coefficient KdV (vKdV) equation first derived in the general case by Grimshaw [13], see also Zhou and Grimshaw [14] and Grimshaw et al. [15,6]. It has the same form as (1) with an extra term,

$$A_t + cA_x + \frac{cQ_x}{2Q}A + \mu AA_x + \lambda A_{xxx} = 0, \quad Q = c^2 I. \tag{7}$$

Here the modal equation depends also on x parametrically, that is $\phi = \phi(z{:}x)$, $c = c(x)$, and hence the coefficients μ, λ, Q also depend (slowly) on x. It is convenient to transform this to the "spatial" evolution form,

$$X = \int^x \frac{dx}{c} - t, \qquad T = \int^x \frac{dx}{c}, \tag{8}$$

$$A_T + \frac{Q_T}{2Q}A + \nu AA_X + \delta A_{XXX} = 0, \tag{9}$$

$$\nu = \frac{\mu}{c}, \qquad \delta = \frac{\lambda}{c^3}. \tag{10}$$

A further simplification is

$$U = Q^{1/2}A, \qquad U_T + \frac{\nu}{Q^{1/2}}UU_X + \delta U_{XXX} = 0. \tag{11}$$

A final transformation yields the canonical form relevant for a polarity change, that is the coefficient α changes sign,

$$U_\tau + \alpha UU_X + U_{XXX} = 0, \tag{12}$$

$$\text{where} \quad \tau = \int^T \delta\, dT, \qquad \alpha = \frac{\nu}{\delta Q^{1/2}}. \tag{13}$$

The coefficient α varies with τ, that is $\alpha = \alpha(\tau)$ in general. This equation has two important conservation laws

$$U_\tau + \left\{\frac{\alpha U^2}{2} + U_{XX}\right\}_X = 0 \tag{14}$$

$$\left\{\frac{U^2}{2}\right\}_\tau + \left\{\frac{\alpha U^3}{3} + UU_{XX} - \frac{U_X^2}{2}\right\}_X = 0 \tag{15}$$

corresponding to conservation of mass and wave action flux respectively. The first arises directly from (12) expressed in flux form, while the second follows from multiplying (12) by U.

Our main interest here is when there is a change of polarity, that is the quadratic coefficient μ in the KdV equation (1) changes sign at a critical point. Since $Q, \lambda \neq 0$ for internal waves, it follows that then α in (12) will likewise change sign. This typically occurs when the pycnocline, a thin layer where the density gradient is very strong, is near the surface in deep water, but near the bottom in shallow water. For mode one waves, it is readily shown that μ is then negative in deep water, but positive in shallow water, and so changes sign as the waves propagate shoreward. The implication is that solitary waves are depression waves in deep water, but elevation waves in shallow water. The behaviour of a solitary wave as it passes through this critical point is now well understood, see the reviews by Grimshaw [5]; Grimshaw et al. [15,6]. As a depression solitary wave approaches a critical point, its amplitude decreases but at the same time a trailing shelf is generated, which grows in amplitude as the critical point is approached. The combination passes through the critical point and then generates a depression rarefaction wave on which rides an undular bore of elevation waves. The corresponding theory for a periodic wave train has only recently been developed by Grimshaw [16] and in contrast, the waves pass through the critical point with only a very small change in amplitude but with a polarity reversal. In this paper we examine how an undular bore behaves as it passes through a critical point, noting that the leading waves in the undular bore are solitary waves while the waves in the rear of the bore are periodic waves. Hence there is an expectation that on passage through the critical point, the front of the undular bore will behave similarly to how a solitary wave behaves, while the rear of the undular bore will behave similarly to how a periodic wave behaves.

In Section 2 we present the Whitham modulation equations for a modulated periodic travelling wave, and describe briefly how these may be used when waves propagate in a region where $\alpha = \alpha(\tau)$ varies. Then in Section 3 we discuss in detail how either a solitary wave, or a periodic wave train, or an undular bore behaves when there is a change of polarity. The analysis is based on the Whitham modulation equations supplemented by some numerical simulations. We conclude in Section 4.

2. Whitham modulation equations

When the coefficient α in (12) is a constant the KdV equation supports a periodic travelling wave, $U(X - V\tau)$, which satisfies the ordinary differential equation,

$$U_X^2 = -\frac{\alpha U^3}{3} + VU^2 + C_1U + C_2, \tag{16}$$

where $C_{1,2}$ are constants of integration. This has the well-known cnoidal wave solution

$$U = a\{b(m) + \mathrm{cn}^2(\gamma\theta; m)\} + d, \quad \theta = k(X - V\tau), \tag{17}$$

$$\text{where} \quad \alpha a = 12m\gamma^2k^2. \qquad b(m) = \frac{1-m}{m} - \frac{E(m)}{mK(m)}, \tag{18}$$

$$\begin{aligned} V - \alpha d &= \frac{\alpha a}{3}\left\{\frac{2-m}{m} - \frac{3E(m)}{mK(m)}\right\} \\ &= 4\gamma^2k^2\left\{2 - m - \frac{3E(m)}{K(m)}\right\}. \end{aligned} \tag{19}$$

Here $cn(x; m)$ is the Jacobian elliptic function of modulus m, $0 < m < 1$, and $K(m)$ and $E(m)$ are the elliptic integrals of the first and second kind,

$$\mathrm{cn}(x; m) = \cos(\phi),$$
$$x = \int_0^\phi \frac{d\phi'}{(1 - m\sin^2\phi')^{1/2}}, \quad 0 \le \phi \le \frac{\pi}{2}, \tag{20}$$

$$\begin{aligned} K(m) &= \int_0^{\pi/2} \frac{d\phi}{(1 - m\sin^2\phi)^{1/2}}, \\ E(m) &= \int_0^{\pi/2} (1 - m\sin^2\phi)^{1/2}\, d\phi. \end{aligned} \tag{21}$$

The expression (17) has period 2π in θ so that $\gamma = K(m)/\pi$, while the spatial period is $2\pi/k$. The (trough-to-crest) amplitude is a and the mean value over one period is d. It is a three-parameter family with parameters k, m, d say. As the modulus $m \to 1$, this becomes a solitary wave, since then $b \to 0$ and $\mathrm{cn}(x) \to \mathrm{sech}(x)$, while $\gamma \to \infty$, $k \to 0$ with $\gamma k = \Gamma$ fixed. As $m \to 0$, $b \to -1/2$, $\gamma \to 1/2$, $\mathrm{cn}(x) \to \cos(x)$, and it reduces to a sinusoidal wave $(a/2)\cos(\theta)$ of small amplitude $a \sim m$ and wavenumber k. The integration constants $C_{1,2}$ can also be expressed in terms of k, m, d but the explicit expressions are not needed here.