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Mechanical balance laws for fully nonlinear and weakly dispersive water waves

Henrik Kalisch [a], Zahra Khorsand [a], Dimitrios Mitsotakis [b,*]

[a] Department of Mathematics, University of Bergen, Norway

[b] School of Mathematics, Statistics and Op. Research, Victoria University of Wellington, New Zealand

HIGHLIGHTS

- Systematic derivation of balance laws for the Serre–Green–Naghdi (SGN) equations.
- Numerical solution of the SGN system using a high-order finite element method.
- Study of the energy balance in undular bores.
- Numerical simulation of shoaling solitary waves.

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This article is dedicated to the memory of Professor Gerald B. Whitham. Professor Whitham's work on dispersive shock waves, modulation theory and the Whitham equation has had a profound impact on the work of the authors, and his encyclopedic book "Linear and Nonlinear Waves" continues to be an important source of inspiration. Much of the authors work on nonlocal dispersive equations, wave breaking and peaking, undular bores and conservation laws has been influenced in some form or another by Professor Whitham's writings

Keywords:
Conservation laws
Serre system
Dispersive shock waves
Solitary waves

ABSTRACT

The Serre–Green–Naghdi system is a coupled, fully nonlinear system of dispersive evolution equations which approximates the full water wave problem. The system is known to describe accurately the wave motion at the surface of an incompressible inviscid fluid in the case when the fluid flow is irrotational and two-dimensional. The system is an extension of the well known shallow-water system to the situation where the waves are long, but not so long that dispersive effects can be neglected. In the current work, the focus is on deriving mass, momentum and energy densities and fluxes associated with the Serre–Green–Naghdi system. These quantities arise from imposing balance equations of the same asymptotic order as the evolution equations. In the case of an even bed, the conservation equations are satisfied exactly by the solutions of the Serre–Green–Naghdi system. The case of variable bathymetry is more complicated, with mass and momentum conservation satisfied exactly, and energy conservation satisfied only in a global sense. In all cases, the quantities found here reduce correctly to the corresponding counterparts in both the Boussinesq and the shallow-water scaling. One consequence of the present analysis is that the energy loss appearing in the shallow-water theory of undular bores is fully compensated by the emergence of oscillations behind the bore front. The situation is analyzed numerically by approximating solutions of the Serre–Green–Naghdi equations using a finite-element discretization coupled with an adaptive Runge–Kutta time integration scheme, and it is found that the energy is indeed conserved nearly to machine precision. As a second application, the shoaling of solitary waves on a plane beach is analyzed. It appears that the Serre–Green–Naghdi equations are capable of predicting both the shape of the free surface and the evolution of kinetic and potential energy with good accuracy in the early stages of shoaling.

1. Introduction

In this paper we study mechanical balance laws for fully nonlinear and dispersive shallow-water waves. In particular, the Serre–Green–Naghdi (SGN) system of equations with variable bathymetry is considered. This system was originally derived for one-dimensional waves over a horizontal bottom in 1953 by F. Serre [1,2]. Several years later, the same system was rederived by Su and Gardner [3]. In 1976, Green and Naghdi [4] derived a two-dimensional fully nonlinear and weakly dispersive system for an uneven bottom which was integrated in one spatial dimension by Seabra-Santos et al. [5] and El et al. [6]. Lannes and Bonneton derived several other systems including the SGN equations using

* Corresponding author. Tel.: +64 44636739; fax: +64 44635045.
E-mail address: dimitrios.mitsotakis@vuw.ac.nz (D. Mitsotakis).
URL: https://sites.google.com/site/dmitsot/ (D. Mitsotakis).

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a new formulation of the water wave problem, [7]. For more information and generalizations of the SGN equations we refer to Lannes [8] and the references therein, while we refer to the paper by Barthélemy [9] for an extensive review.

The Serre–Green–Naghdi (SGN) system and several variants of it are extensively used in coastal modeling [10–12,8]. In the present contribution, the focus is on the derivation and use of associated mechanical balance equations, and in particular a differential energy balance equation. While it is known that the equations admit local conservation equations corresponding to mass, momentum and energy conservation if the bed is even [13], it appears that the connection to the mechanical balance laws of the original Euler equations has not been firmly established so far. One possible method for establishing the link between the conservation laws and the requisite physical quantities is outlined in the work of Miles and Salmon [14]. In this work, the Serre–Green–Naghdi (SGN) equations are shown to follow from Hamilton's principle of least action in the same way as the full free-surface water wave problem does if it is assumed that the fluid moves in vertical columns, or in other words that the horizontal displacement of fluid particles is uniform throughout the fluid column. This approximation preserves several of the symmetries of the full water-wave problem [15], and in particular gives rise to corresponding conservation laws for mass, momentum and energy through the use of Noether's theorem.

We follow a different route in that we make the same approximation in both the evolution equations (Euler equations) and in the corresponding mechanical balance laws directly. Using this approach, we show that the first three conservation laws of the Serre–Green–Naghdi (SGN) equations arise as approximations of mechanical balance laws in the context of the Euler equations, both in the case of even beds, and in the case of nontrivial bathymetry. While one may have doubts about the link between the resulting approximate balance laws at a mathematical level, it can be established (see [13]) that these balance equations also arise as exact consequences of the Serre–Green–Naghdi (SGN) equations.

As it was shown in [16], the Serre–Green–Naghdi (SGN) equations also admit a fourth conservation law which may be interpreted as conservation of potential vorticity, and arises from a certain relabeling symmetry of the Lagrangian density used in [14]. This fourth conservation law can also be shown to be related to a kinematic identity similar to Kelvin's circulation theorem [17].

Let us first review some modeling issues regarding the Serre–Green–Naghdi (SGN) system. Suppose a denotes a typical amplitude, and l a typical wavelength of a wavefield under study. Suppose also that b_0 represents the average water depth. In order to be a valid description of such a situation, the SGN equations require the shallow water condition, $\beta \doteq b_0^2/l^2 \ll 1$. In contrast, the range of validity of the weakly nonlinear and weakly dispersive Boussinesq equations is limited to waves with small amplitude and large wavelength, i.e. $\alpha \doteq a/b_0 \ll 1$ and $\beta \ll 1$. In this scaling regime, one also finds the weakly nonlinear, fully dispersive Whitham equation [8,18,19].

The SGN equations can be derived by depth-averaging the Euler equations and truncating the resulting set of equations at $\mathcal{O}(\beta^2)$ without making any assumptions on the order of α, other than $\alpha \le \mathcal{O}(1)$.

In their dimensionless and scaled form the SGN equations can be written as

$$\eta_t + [h\bar{u}]_x = 0, \tag{1a}$$

$$\bar{u}_t + \bar{u}\bar{u}_x + g\eta_x + \frac{1}{h}\left[h^2\left(\frac{1}{3}\mathcal{P} + \frac{1}{2}\mathcal{Q}\right)\right]_x - b_x\left(\frac{1}{2}\mathcal{P} + \mathcal{Q}\right) = 0, \tag{1b}$$

with $\mathcal{P} = h\left[\bar{u}_x^2 - \bar{u}_{xt} - \bar{u}\bar{u}_{xx}\right]$ and $\mathcal{Q} = -b_x(\bar{u}_t + \bar{u}\bar{u}_x) - b_{xx}\bar{u}^2$, $x \in \mathbb{R}, t > 0$, along with the initial conditions $h(x,0) = h_0(x)$, $\bar{u}(x,0) = \bar{u}_0(x)$. Here, $\eta = \eta(x,t)$ is the free surface displacement, while

$$h \doteq \eta + b, \tag{2}$$

denotes the total fluid depth. The unknown $\bar{u} = \bar{u}(x,t)$ is the depth-averaged horizontal velocity, and η_0, $\bar{u}_0$ are given real functions, such that $\eta_0 + b > 0$ for all $x \in \mathbb{R}$. In these variables, the location of the horizontal bottom is given by $z = -b$ (cf. Fig. A.1). For a review of the derivation and the basic properties of this system we also refer to [9,20].

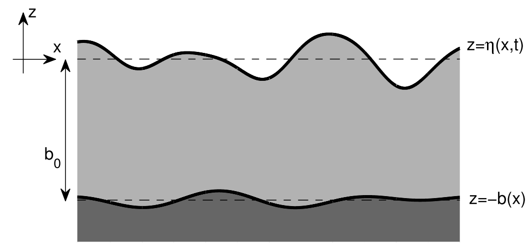

Fig. A.1. The geometry of the problem.

In the case of small-amplitude waves, i.e. if $\beta \sim \alpha$, the SGN equations reduce to Peregrine's system [21]. On the other hand, in the case of very long waves, i.e. $\beta \to 0$, the dispersive terms disappear, and the system reduces to the nondispersive shallow water equations.

The SGN system for waves over a flat bottom possesses solitary and cnoidal wave solutions given in closed form. For example, the solitary wave with speed c_s can be written as

$$h_s(\xi) \doteq h_s(x,t) = a_0 + a_1 \operatorname{sech}^2(K_s\,\xi), \tag{3a}$$

$$u_s(\xi) \doteq u_s(x,t) = c_s\left(1 - \frac{a_0}{h_s(\xi)}\right), \tag{3b}$$

where $\xi = x - c_s t$, $K_s = \sqrt{3a_1/4a_0^2c_s^2}$, $c_s = \sqrt{a_0 + a_1}$, and $a_0 > 0$ and $a_1 > 0$. For more information about the solitary and cnoidal waves and their dynamical properties we refer to [9,22–26].

It is important to note that the SGN system has a Hamiltonian structure, even in the case of two-dimensional waves over an uneven bed cf. [24,27–29]. Specifically, any solution $(h, \bar{u})$ of (1) conserves the Hamiltonian functional

$$\mathcal{H}(t) = \frac{1}{2}\int_{-\infty}^{\infty} g\eta^2 + h\bar{u}^2 - h\left[h_x b_x + \frac{1}{2}hb_{xx} - b_x^2\right]\bar{u}^2 - \frac{1}{3}\left[h^3\bar{u}_x\right]_x \bar{u}\,dx, \tag{4}$$

in the sense that $d\mathcal{H}(t)/dt = 0$. Note however that (1a), (1b) are recovered only if a non-canonical symplectic structure matrix is used. While in many simplified models equations, the Hamiltonian functions does not represent the mechanical energy of the wave system [30], in the case of SGN, the Hamiltonian does represent the approximate total energy of the wave system. Thus the Hamiltonian can be written in the form

$$\mathcal{H}(t) = \int_{-\infty}^{\infty} E(x,t)\,dx,$$

where the integrand

$$E = \frac{1}{2}\left(g\eta^2 + h\bar{u}^2 - h\left[h_x b_x + \frac{1}{2}hb_{xx} - b_x^2\right]\bar{u}^2 - \frac{1}{3}\left[h^3\bar{u}_x\right]_x\bar{u}\right)$$

is the depth-integrated energy density. In the present paper, we also identify a local depth-integrated energy flux q_E, such that an equation of the form

$$\frac{\partial E}{\partial t} + \frac{\partial q_E}{\partial x} = 0, \tag{5}$$