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Interaction of solitons with long waves in a rotating fluid

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HIGHLIGHTS

- Interaction of a KdV soliton with a long wave is studied in a rotating ocean.
- Long background waves are sinusoidal wave and periodic sequence of parabolic arcs.
- The model dynamical system is derived and studied analytically and numerically.
- Solitons riding on long wave can propagate on long distances in a rotating ocean.

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ABSTRACT

Interaction of a soliton with long background waves is studied within the framework of rotation modified Korteweg–de Vries (rKdV) equation. Using the asymptotic method for solitons propagating in the field of a long background wave we derive a set of ODEs describing soliton amplitude and phase with respect to the background wave. The shape of the background wave may range from a sinusoid to the limiting profile representing a periodic sequence of parabolic arcs. We analyse energy exchange between a soliton and the long wave taking radiation losses into account. It is shown that the losses can be compensated by energy pumping from the long wave and, as the result, a stationary soliton can exist, unlike the case when there is no variable background. A more complex case when a free long wave attenuates due to the energy consumption by a soliton is also considered. Some of the analytical results are compared with the results of direct numerical calculations within the framework of the rKdV equation.

1. Introduction

In this paper we study the interaction of a solitary wave with a long background wave within the framework of the so called rotation modified Korteweg–de Vries (rKdV) equation. This equation was derived in 1978 [1] as the model describing long surface and internal waves in rotating oceans. In the dimensional variables the equation reads

$$\frac{\partial}{\partial x}\left(\frac{\partial \upsilon}{\partial t}+c_0\frac{\partial \upsilon}{\partial x}+\alpha\upsilon\frac{\partial \upsilon}{\partial x}+\beta\frac{\partial^3 \upsilon}{\partial x^3}\right)=\gamma\upsilon, \quad (1)$$

where the coefficients c_0, α, β, and γ depend on the environmental parameters (water depth, stratification, etc.). This equation can be considered as the generalisation of the classical Korteweg–de Vries (KdV) equation, which follows from Eq. (1) when $\gamma = 0$.

Later on it was realised that Eq. (1) is fairly general since it combines the effects of small quadratic nonlinearity and two types of dispersion—the small-scale dispersion, proportional to β, and large-scale dispersion, proportional to γ. Similar equations were derived for the description of weakly nonlinear waves in random media [2], magnetosonic waves in rotating plasma [3], waves in relaxing media [4], electromagnetic waves in nonlinear transmission lines [5], and strain waves in elastic bi-layers [6].

Eq. (1) is, apparently, non-integrable, and even its stationary solutions are not known thus far in the analytic form. Particular numerical and approximate solutions were constructed in many publications, see, for instance, [3,5,7–13]. Numerical studies have shown that in some cases stationary solutions can be interpreted as a superposition of a long periodic background wave described

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approximately by Eq. (1) with $\beta = 0$, and a solitary wave approximately representing KdV soliton which is a solution of Eq. (1) with $\gamma = 0$; the examples are presented in [5,7,10,12,13]. In such cases a soliton can ride on the crest or on the trough of the long background wave. In the meantime, if the soliton amplitude is not properly matched with the amplitude of the background wave, the soliton travels along the long wave periodically accelerating and decelerating, growing and decaying [7].

One of the intriguing features of solitary wave dynamics within the framework of rKdV equation is related to the fact that solitary waves cannot exist within the framework of this equation with $\beta\gamma > 0$ which is practically always the case for oceanic waves. This is the result of the rigorous "antisoliton theorem" proven in [14,15] and then in many other papers. However, as follows from the aforementioned numerical results, solitary waves may exist on a long background wave.

In what follows we present the asymptotic theory describing the dynamics of a KdV soliton on the long background wave. The background wave is taken as one of the particular solutions of the reduced rKdV equation (1) with $\beta = 0$. As shown in [1,5,10,16,17], there exists a family of exact periodic solutions of the reduced rKdV equation. After giving general relationships in Section 2, we consider soliton interaction with two limiting representatives of the family of periodic solutions, viz., the sinusoidal wave (Section 3) and the periodic sequence of parabolic arcs (Section 4). Then, in Section 5 we present the results of direct numerical simulation of soliton interaction with a periodic background wave and discuss the results obtained in Section 6.

2. Interaction of a soliton with a long background wave

Solutions of the rKdV equation (1) essentially depend on the sign of the dispersive coefficients β and γ. As mentioned, for $\beta\gamma > 0$ (the "oceanic case"), solitary waves with zero asymptotics at the infinity do not exist. In the opposite case $\beta\gamma < 0$ (describing, e.g., magnetosonic waves in a rotating plasma [3] and internal waves in a rotating ocean with shear flows [18]) the "antisoliton theorem" is not valid, and solitary waves can exist. Their structure has been investigated in [3]. In this paper we will consider only the former case when $\beta\gamma > 0$.

Although the rKdV equation (1) is, apparently, nonintegrable, it possesses many integrals of motion (see, e.g., [10] and references therein). Here we will use one of them, the "zero mass" integral:

$$M \equiv \int \upsilon(x,t)dx = 0. \tag{2}$$

The integration here is taken either over the wave period for periodic waves or over the entire axis x for localised solutions. Note that in many other cases, including the KdV equation, the "wave mass" M can be an arbitrary constant which is determined by initial conditions. In the case of rKdV equation, Eq. (2) is not just the integral of motion, but rather a constraint which demands that initial conditions must be consistent with the zero-mass condition. It should be pointed out, however, that the condition (2) may be violated for singular solutions—for details see [17].

To achieve more universality, it is constructive to reduce Eq. (1) to the dimensionless form. By means of transformation

$$x' = \left(\frac{\gamma}{\beta}\right)^{1/4}(x - c_0 t), \qquad t' = \beta\left(\frac{\gamma}{\beta}\right)^{3/4} t,$$
$$u = \frac{\alpha}{\beta}\left(\frac{\beta}{\gamma}\right)^{1/2}\upsilon. \tag{3}$$

Eq. (1) can be reduced to (the primes in new variables x and t are further omitted):

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) = u. \tag{4}$$

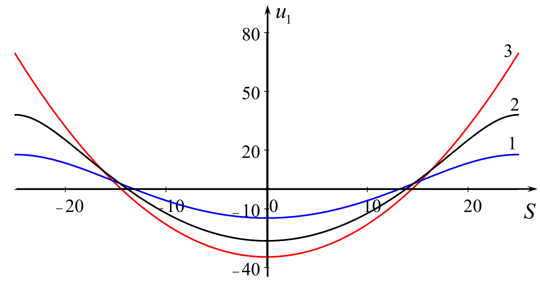

Fig. 1. (Colour online) A family of zero-mass stationary periodic solutions of the reduced rKdV equation (4) with $\Lambda = 50$, when the third-order derivative is omitted. Line 1 pertains to quasi-sinusoidal wave with $c = 64$, line 2 illustrates a nonlinear wave with $c = 66$, and line 3 represents a periodic sequence of parabolic arcs with $c = (25/3)^2 \approx 69.4$.

We seek for a solution to this equation in the form $u(t,x) = u_1(t,x) + u_2(t,x)$, where $u_1(t,x)$ is a smooth periodic background wave with the wavelength Λ, and $u_2(t,x)$ is a KdV soliton with slowly varying amplitude and width (see below). As shown in the Appendix, if the soliton width is much smaller than the wavelength of the background wave, the equations for these functions can be separated. First we assume that the function u_1 is given and it represents a particular stationary solution to the reduced Eq. (4) in which the third-order derivative responsible for the small-scale dispersion is omitted, i.e., $u_2 = u_2(s = x - ct)$, where c is a constant wave speed. This solution satisfies the equation

$$\frac{d^2}{ds^2}\left(\frac{1}{2}u_1^2 - cu_1\right) = u_1. \tag{5}$$

The shape of this wave can vary from the small amplitude sinusoidal wave to the limiting periodic wave in the form of a sequence of parabolic arcs [1,5,10,16] (see Fig. 1); all waves of this family have zero mean value.

As mentioned above, the term u_2 in the trial solution describes a narrow KdV soliton with the characteristic width $\Delta \ll \Lambda$, which satisfies Eq. (4) with zero right-hand side. As a result, we have the following equation for the solution u_2 (for details see Appendix):

$$\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial x} + \frac{\partial^3 u_2}{\partial x^3} \approx \int u_2 dx - \left(u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_1}{\partial x}\right) \tag{6}$$

where u_1 is given as a solution of Eq. (5). Here we neglect the inverse impact of a soliton on the long background wave in Eq. (5); this effect will be discussed later in this paper. At the same time, the parameters of a solitary wave may vary in space and time depending on its position on the background wave.

Since, as mentioned, the characteristic soliton width is assumed small, the background wave and its spatial derivative can be considered locally constant in the vicinity of soliton maximum. This allows one to seek a solution to Eq. (6) for u_2 in the form of a KdV soliton with the parameters slowly varying in time: amplitude $A(t)$, width $\Delta(t)$, and 'phase' $S(t)$:

$$u_2 = A\,\mathrm{sech}^2\frac{\zeta - S}{\Delta} - p, \tag{7}$$

where the phase S is the soliton peak position with respect to a certain point of the background wave profile, $s = S$, and

$$\zeta = x - \int_0^t V\,dt, \qquad V = \frac{A}{3} - p + u_1(S),$$
$$\Delta = \sqrt{\frac{12}{A}}, \qquad p = \frac{4\sqrt{3A}}{\Lambda}. \tag{8}$$

In Eq. (7) p is a small negative pedestal which is added to satisfy the zero-mass condition (2) on the interval $[0, \Lambda]$. However, within