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Dispersive shock waves in nematic liquid crystals

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HIGHLIGHTS

- Solution for an undular bore in a nematic liquid crystal derived.
- Undular bore found to be of Korteweg–de Vries type.
- Reasons for Korteweg–de Vries bore detailed.
- Excellent agreement with numerical solutions found.
- Undular bore solution related to previous experimental results.

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ABSTRACT

The propagation of coherent light with an initial step intensity profile in a nematic liquid crystal is studied using modulation theory. The propagation of light in a nematic liquid crystal is governed by a coupled system consisting of a nonlinear Schrödinger equation for the light beam and an elliptic equation for the medium response. In general, the intensity step breaks up into a dispersive shock wave, or undular bore, and an expansion fan. In the experimental parameter regime for which the nematic response is highly nonlocal, this nematic bore is found to differ substantially from the standard defocusing nonlinear Schrödinger equation structure due to the effect of the nonlocality of the nematic medium. It is found that the undular bore is of Korteweg–de Vries equation-type, consisting of bright waves, rather than of nonlinear Schrödinger equation-type, consisting of dark waves. In addition, ahead of this Korteweg–de Vries bore there can be a uniform wavetrain with a short front which brings the solution down to the initial level ahead. It is found that this uniform wavetrain does not exist if the initial jump is below a critical value. Analytical solutions for the various parts of the nematic bore are found, with emphasis on the role of the nonlocality of the nematic medium in shaping this structure. Excellent agreement between full numerical solutions of the governing nematicon equations and these analytical solutions is found.

1. Introduction

Solitary waves, or solitons for integrable equations, are thought of as the defining solution of many nonlinear wave equations, such as the Korteweg–de Vries (KdV) equation, the nonlinear Schrödinger (NLS) equation and the Sine-Gordon equation [1]. However, such equations also possess a generic solution which is just as characteristic as the soliton solution, which arises in many applications and is just as widely observed. This solution is the undular bore, or dispersive shock wave. The term undular bore arises from their first observation as wave structures in fluids and as this is the first name applied, it will be used in this work, rather than the term dispersive shock wave. For nonlinear waves governed by dispersive equations, undular bores arise when an initial jump, or near jump, linking two levels is smoothed by the action of dispersion, resulting in a smooth wavetrain linking these two levels. The generic structure of an undular bore is that it has one edge consisting of solitary waves with the opposite edge consisting of linear, dispersive waves. An undular bore is the dispersive equivalent of a gas dynamic shock, for which viscous effects smooth out the jump [1], as opposed to the dispersive effects smoothing out an undular bore. For this reason, an undular bore is also termed a dispersive shock wave. An undular bore is a non-steady wavetrain which continually expands in length. It should be noted that there is another, steady bore arising in water wave theory, a viscous bore [1,2]. This bore is steady due to the effect of viscosity. This work will deal with undular bores in a nonlinear optical system, for which there is no equivalent of fluid viscosity. Therefore, the bores dealt with in the present work are all undular bores.

As a viscous bore is steady, it is relatively straightforward to obtain a solution for it [3]. The unsteady nature of an undular

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bore made finding a solution for it more difficult. Whitham [1,4–6] developed modulation theory, or the method of averaged Lagrangians, as a method to analyse dispersive wavetrains slowly varying in both space and time. This method is related to the method of multiple scales in perturbation theory [7]. In particular, Whitham derived the modulation equations for the KdV equation [1,5]. As the periodic (cnoidal) wave solutions of the KdV equation are stable, these modulation equations form a hyperbolic system for the parameters of the modulated cnoidal wave. Furthermore, these modulation equations could be set in Riemann invariant form. It was subsequently realised that a simple wave solution of these hyperbolic modulation equations corresponds to an undular bore solution of the KdV equation [8,9]. A major advance occurred when it was shown using functional analysis that the ability to set the modulation equations for the KdV equation in Riemann invariant form was linked to the KdV equation having an inverse scattering solution [10], which meant that it was then clear how to calculate the modulation equations for other nonlinear dispersive wave equations having an inverse scattering solution. As few nonlinear dispersive wave equations have an inverse scattering solution, the utility of using Whitham modulation theory to find undular bore solutions of nonlinear dispersive wave equations was greatly extended when a method was found to determine the leading and trailing edges of an undular bore in the absence of the full modulation equations and for equations for which there is no inverse scattering solution to enable the modulation equations to be found using standard techniques [11,12]. Bore solutions for a variety of nonlinear dispersive equations have now found use in a wide range of physical applications, for example water waves [13–15], oceanography [16], meteorology [17–19], geophysics [20–23] and nonlinear optics [24–28].

The present work is concerned with determining the undular bore solution for the equations governing a specific class of nonlinear optical media, the nematic liquid crystal [29–31] in the defocusing regime [32] for the usual experimental parameter range. Nematic liquid crystals are a nonlinear optical medium which supports solitary waves [30,31,33], termed nematicons. They are usually a focusing medium, so that the nematicons are bright waves and the resulting modulation equations for the nematic equations are elliptic and so do not have an undular bore solution. However, nematic liquid crystals can be made defocusing through the addition of azo-dyes [32]. The original applications of undular bores were in fluid mechanics and water waves, hence their name, but they have recently found extensive application in nonlinear optics. Both experimental and numerical investigations have shown that undular bores can be generated in thermal nonlinear optical media [34–37] and nonlinear crystals [38,39], among other defocusing nonlinear optical media. As these undular bores form in defocusing media, the bores are dark bores, that is dips in a background carrier wave. Nematic liquid crystals are termed nonlocal media as in the usual experimental regime the elastic response of the nematic to an optical beam extends far beyond the beam [30,31]. While structures with some resemblance to undular bores can form in focusing nonlinear optical media [34,39], such as nematic liquid crystals, and some approximate analytical theory has been developed for these [40,41], there has been no theory developed for undular bores in defocusing nonlinear, nonlocal media, such as nematic liquid crystals and thermal media, which is valid for experimental parameter ranges. In this context the equations governing nonlinear beams in defocusing thermal nonlinear media [34–37] are the same as those governing nonlinear beams in defocusing nematic liquid crystals [30,31]. In the so-called local limit the equations for optical beams in a nematic liquid crystal reduce to the standard NLS equation, for which there is an known undular bore solution [42], but, as stated, this is not the usual experimental regime.

In the present work, the undular bore solution for the equations governing nonlinear optical beam propagation in a defocusing nematic liquid crystal will be developed. It is found that the undular bore is of KdV-type, even though the equation governing the electric field of the light beam is of NLS-type. The bore then consists of bright waves, rises above a background level, rather than the dark waves, dips in a background level, of a defocusing NLS bore [42]. The method of El [11,12] is used to show the non-existence of an NLS-type bore in the nonlocal limit. For an initial light intensity jump above a critical height, ahead of the KdV bore is a uniform wavetrain with a short front which brings the solution down to the initial level ahead. This wavetrain is generated by a phase mis-match between the KdV bore and the initial state. A phase and group velocity argument is used to find the leading and trailing edges of this uniform wavetrain. This argument predicts a minimum jump height for the uniform wavetrain to exist, which is confirmed by numerical solutions. Outside of the KdV bore and uniform wavetrain regions, the solution is given by the non-dispersive limit of the nematic equations. These various parts of the analytical solution for the nematic bore are compared with full numerical solutions of the governing equations and good to excellent agreement is found, depending on the initial jump height.

Previous experimental [34] and numerical [34–37] studies of undular bores in defocusing nonlinear, nonlocal thermal media were for $O(1)$ or $O(10)$ values of the nonlocality parameter, and so were not in the highly nonlocal regime typical of nematic liquid crystals for which the nonlocality parameter is $O(100)$, and were generated by gradient catastrophes of finite initial conditions. True undular bores cannot be generated from a finite initial condition as true bores require the continual generation of waves as the bore spreads. Breaking finite initial conditions give an approximation to an undular bore for finite propagation distances as the inverse scattering solution of the NLS equation, both focusing and defocusing, shows that a finite initial condition generates a finite number of solitons plus dispersive radiation [1]. However, some of these experimental and numerical results show evidence of the KdV-bore type structure found in the present work [34,36], as will be discussed in more detail below.

2. Dark nematicon equations

Let us consider the propagation of polarised coherent light through a cell filled with a nematic liquid crystal. The light is taken to propagate in the z direction and the x direction is taken as the direction of polarisation of the electric field of the light. The light beam is taken to be $(1 + 1)$ dimensional, which is a valid approximation for light in a nematic liquid crystal cell due to the widely different aspect ratios of the cell in the directions transverse to propagation [43]. To overcome the optical Freédericksz threshold [29], an external low frequency electric field is applied in the polarisation direction to pre-tilt the nematic molecules [33]. With this pre-tilting field, optical solitary waves in the nematic liquid crystal, termed nematicons [30,31], and other nonlinear optical waves can be formed with milliwatt power beams. If the beam power is high, undesirable physical effects can occur, even to the extent of the nematic phase becoming unstable [33]. Nematic liquid crystals usually form a focusing medium, so that they support bright optical solitary waves, bright nematicons [30,31]. However, a nematic liquid crystal can become a defocusing medium through the addition of azo-dyes [32], so that dark optical solitary waves, dark nematicons, can be supported. In the paraxial, slowly varying envelope approximation, the non-dimensional equations governing the propagation of the optical beam through the defocusing nematic liquid crystal are [30–32,44]

$$i\frac{\partial u}{\partial z} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - 2\theta u = 0, \tag{1}$$