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Blow up criterion of strong solution for 3D viscous liquid–gas two-phase flow model with vacuum

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HIGHLIGHTS

- We establish a blow-up criterion of the strong solution to the viscous liquid–gas two-phase flow model.
- The criterion is only in terms of the divergence of the velocity field.
- The initial vacuum is allowed.
- There is no extra restriction on viscosity coefficients.
- Both the Cauchy problem and initial–boundary value problem are considered.

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ABSTRACT

In this paper, we establish a blow-up criterion to the local strong solution to the three dimensional (3D) viscous liquid–gas two-phase flow model only in terms of the divergence of the velocity field. Moreover, the initial vacuum is allowed, and there is no extra restriction on viscous coefficients. Both the Cauchy problem and initial-boundary value problem are considered in this paper.

1. Introduction

In this paper, we concern the 3D viscous liquid–gas two-phase flow model

$$\begin{cases} m_t + \operatorname{div}(mu) = 0, & \text{in } \Omega \times (0,T), \\ n_t + \operatorname{div}(nu) = 0, & \text{in } \Omega \times (0,T), \\ (mu)_t + \operatorname{div}(mu \otimes u) + \nabla P & \\ \quad = \mu \Delta u + (\lambda + \mu)\nabla \operatorname{div} u, & \text{in } \Omega \times (0,T), \end{cases} \tag{1}$$

with the initial conditions

$$(m, n, u)|_{t=0} = (m_0, n_0, u_0) \quad \text{in } \Omega, \tag{2}$$

and two types of boundary conditions:

(1) Dirichlet boundary condition:

$$u = 0 \quad \text{on } \partial\Omega, \tag{3}$$

for $\Omega \subset \mathbb{R}^3$ being a bounded smooth domain;

(2) Cauchy problem:

$$u(x,t) \to 0, \qquad (m,n)(x,t) \to (0,0), \quad \text{as } |x| \to \infty, \tag{4}$$

for $\Omega = \mathbb{R}^3$.

The unknown variables $m = \alpha_l \rho_l$, $n = \alpha_g \rho_g$, $u = (u_1, u_2, u_3)$ and $P = P(m, n)$ denote the liquid mass, gas mass, the velocity of the fluid, and the common pressure for both phases, respectively. α_l and $\alpha_g \in [0, 1]$ denote the liquid and gas volume fractions, respectively, with the relation

$$\alpha_l + \alpha_g = 1. \tag{5}$$

ρ_l and ρ_g denote the liquid and gas densities, respectively. μ and λ are viscosity constants, with the physical conditions

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0. \tag{6}$$

Moreover, the pressure satisfies the equations of state

$$P = P_{l,0} + a_l^2(\rho_l - \rho_{l,0}) \quad \text{and} \quad P = \rho_g a_g^2, \tag{7}$$

where a_l and a_g are sonic speeds in the liquid and gas, respectively; $P_{l,0}$ and $\rho_{l,0}$ are the reference pressure and density given as

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constants. In view of the relationship (5) and (7), the pressure satisfies

$$P^2 + \left(a_l^2 \rho_{l,0} - P_{l,0} - n a_g^2 - m a_l^2\right) P - \left(a_g^2 a_l^2 \rho_{l,0} - a_g^2 P_{l,0}\right) n = 0,$$

which implies

$$P(m,n) = \frac{1}{2} a_l^2 \left(-b(m,n) + \sqrt{b(m,n)^2 + c(m,n)}\right), \tag{8}$$

here

$$b(m,n) = k_0 - m - \left(\frac{a_g}{a_l}\right)^2 n, \qquad c(m,n) = 4k_0 \left(\frac{a_g}{a_l}\right)^2 n,$$

and

$$k_0 = \rho_{l,0} - \frac{P_{l,0}}{a_l^2} > 0.$$

In this paper, we study a simplified model (1) of Drift-flux type, in which we suppose that the two fluids have the common pressure and the same velocity, while the external force and the effect of gas in the convective term can be ignored in the mixture momentum equation. At the beginning, we recall the original form of the model (1), as follows,

$$\begin{cases} (\alpha_l \rho_l)_t + \operatorname{div}(\alpha_l \rho_l u_l) = 0, \\ (\alpha_g \rho_g)_t + \operatorname{div}(\alpha_g \rho_g u_g) = 0, \\ (\alpha_l \rho_l u_l)_t + \operatorname{div}(\alpha_l \rho_l u_l \otimes u_l) + \alpha_l \nabla P(\alpha_l \rho_l, \alpha_g \rho_g) \\ \qquad + \Delta P(\alpha_l \rho_l, \alpha_g \rho_g) \nabla \alpha_l \\ \quad = Q_l + M_l + \operatorname{div}(\mu \alpha_l (\nabla u_l + \nabla u_l^T)) + \nabla(\lambda \alpha_l \operatorname{div} u_l), \\ (\alpha_g \rho_g u_g)_t + \operatorname{div}(\alpha_g \rho_g u_g \otimes u_g) + \alpha_g \nabla P(\alpha_l \rho_l, \alpha_g \rho_g) \\ \qquad + \Delta P(\alpha_l \rho_l, \alpha_g \rho_g) \nabla \alpha_g \\ \quad = Q_g + M_g + \operatorname{div}(\mu \alpha_g (\nabla u_g + \nabla u_g^T)) + \nabla(\lambda \alpha_g \operatorname{div} u_g), \end{cases} \tag{9}$$

where ΔP is the correction term, M_g and M_l represent interface modeling interactions between the phases, and satisfy

$$M_g + M_l = 0.$$

Q_g and Q_l represent external forces (friction and gravity) on gas flows and liquid flows, respectively. Summing the momentum equations in (9) yields directly

$$\begin{aligned} &(\alpha_l \rho_l u_l + \alpha_g \rho_g u_g)_t + \operatorname{div}(\alpha_l \rho_l u_l \otimes u_l \\ &\quad + \alpha_g \rho_g u_g \otimes u_g) + \nabla P(\alpha_l \rho_l, \alpha_g \rho_g) \\ &= Q_l + Q_g + \mu \Delta(\alpha_l u_l + \alpha_g u_g) + (\mu + \lambda) \nabla \operatorname{div}(\alpha_l u_l + \alpha_g u_g). \end{aligned}$$

Furthermore, neglecting the external forces and assuming the gas and liquid flows possess the consistent velocity, we obtain the simplified model (1).

In the following, we would like to recall some known results about the viscous liquid–gas two-phase flow model. In one dimensional case, when the fluids connected to vacuum state discontinuously, Evje and Karlsen [1] first studied the existence and uniqueness of the global weak solution to the free boundary value problem with the viscous coefficient $\mu(m) = k_1 \frac{m^\beta}{(\rho_l - m)^{\beta+1}}$, $\beta \in \left(0, \frac{1}{3}\right)$. Moreover, the asymptotic behavior and the regularity of the solution has been considered in [2,3]. For more properties on the 1D model (1) or related model, we can refer [4–9] to the readers. For the multi-dimensional case, many problems such as the regularity of the solution are still totally open. Guo and etc. showed the existence of the global weak solution with initial vacuum in [10] for sufficiently small initial energy. Later on, Wen and etc. [11] established the existence and uniqueness of the local strong solution to the 3D system (1) with initial vacuum, moreover, they gave a blow-up criterion in terms of the estimate of $\|m\|_{L^\infty(0,T;L^\infty)}$ for the strong solution with vacuum, with the additional restriction on the viscosity coefficients

$$\frac{25\mu}{3} > \lambda.$$

Moreover, Hou and Wen [12] obtained a blow up criterion with the estimate of $L_t^1 L_x^\infty$ norm of the deformation tensor of the velocity gradient $\mathcal{D}(t) = \frac{1}{2}(\nabla u + \nabla u^T)$ with vacuum. Yao etc. in [13] established a blow-up criterion only in terms of the gradient of velocity field for the strong solution to 3D case in bounded domain, provided that the initial vacuum is absent. Namely, suppose that $T^* < \infty$ is maximal existence time to the strong solution, then

$$\lim_{T \to T^*} \|\nabla u\|_{L^1(0,T;L^\infty)} = \infty. \tag{10}$$

The methods mentioned in the above works are borrowed from the ideas in a series works [14–18] on compressible Navier–Stokes equations. Next, we will introduce some similar results on blow-up criteria to the compressible Navier–Stokes equations. For the 3D compressible Navier–Stokes equations, Sun, Wang and Zhang [18] established a blow-up criterion about the upper bound of density for strong solution, with the initial vacuum in both bounded smooth domain and $\mathbb{R}^3$, provided the viscosity coefficients satisfy the additional restriction $\lambda < 7\mu$. Huang and Xin [17] obtained a following blow-up criterion under the above viscosity coefficients restriction, i.e., if $T^* < \infty$ is the maximal time of the existence of the classical solution, that

$$\lim_{T \to T^*} \int_0^T \|\nabla u\|_{L^\infty} dt = \infty,$$

when there is initial vacuum. Later, Huang and etc. in [16] removed the restriction $\lambda < 7\mu$ and gave the blow-up criterion as

$$\lim_{T \to T^*} \int_0^T \|\mathcal{D}u(t)\|_{L^\infty} dt = \infty,$$

where $\mathcal{D}(t) = \frac{1}{2}(\nabla u + \nabla u^T)$.

Recently, for the 3D compressible Navier–Stokes equations, Du and Wang [19] improved the results and showed that

$$\lim_{T \to T^*} \int_0^T \|\operatorname{div} u(t)\|_{L^\infty}^2 dt = \infty,$$

which implies that the divergence of the velocity field plays the dominant role in the blowup mechanism instead of the gradient tensor or its symmetry part of the velocity field.

Therefore, motivated by these works, for the 3D viscous liquid–gas two-phase flow model (1), we expect to establish a similar blow-up criterion only in terms of divergence of velocity field as in [19], instead of (10), which is the main purpose in this paper. We also give the rigorous proof of this assertion for the initial–boundary value problem (1)–(3) and the Cauchy problem (1), (2), (4).

Throughout this paper, we denote the simple notation as

$$\int f dx = \int_\Omega f dx.$$

For $1 \le p \le \infty$ and integer $k \ge 0$, we denote the standard Lebesgue and Sobolev spaces as follows,

$$\begin{cases} L^p = L^p(\Omega), \quad D^{k,p} = \{u \in L^1_{loc}(\Omega) : \|\nabla^k u\|_{L^p} < \infty\}, \\ W^{k,p} = L^p \cap D^{k,p}, \quad D^k = D^{k,2}, \quad H^k = W^{k,2}, \\ D_0^1 = \{u \in L^6 : \|\nabla u\|_{L^2} < \infty, \text{and (3) or (4) holds}\}, \\ H_0^1 = L^2 \cap D_0^1, \quad \|u\|_{D^{k,p}} = \|\nabla u\|_{L^p}, \quad Q_T = \bar{\Omega} \times [0,T]. \end{cases}$$

In the following, we give the definition of the strong solution and the local existence of the unique strong solution for the viscous liquid–gas two-phase flow model (1) with vacuum for smooth bounded domain, which has been established in [11]. It can be extended to Cauchy problem without any additional difficulty, and we omit it here.